Naïve Infinitesimal Analysis

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Abstract

This research has been done with the aim of building a new understanding of the nonstandard mathematical analysis. In order to achieve this goal, we first explore the two existing models of numbers: reals ($\mathbb{R}$) and hyperreals ($^*\mathbb{R}$), and also the main feature of the latter one which is the transfer principle – which, as Goldblatt said, is where the strength of nonstandard analysis lies. However, here we analyse some serious problems with that transfer principle and moreover, propose an idea to solve it: combining the two languages of $\mathbb{R}$ and $\mathbb{R}^{\mathbb{Z}_<}$ into one language. Nevertheless, there is one obvious big problem with this idea, which is the occurrence of contradiction.

Two possible ways are proposed in resolving this contradiction issue. One of them, which we favour in this research, is by having a subsystem in our new theory. This idea was based on Chunk and Permeate strategy proposed by Brown and Priest in 2003. In the process of doing that, we then turn to the primary contribution of this thesis: the construction of a new set of numbers, $\mathbb{R}^{\mathbb{Z}_<}$, which also include infinities and infinitesimals in it. The construction of this new set is done naïvely (in comparison to other sets) in the sense that it does not require any heavy mathematical machinery and so it will be much less problematic in a long term. Despite of its naïve way of construction, it has been demonstrated in this thesis that the set $\mathbb{R}^{\mathbb{Z}_<}$ is still a robust and rewarding set to work in. We further develop some analysis and topological properties of $\mathbb{R}^{\mathbb{Z}_<}$, where not only

we recover most of the basic theories that we have classically, but we also introduce some new enthralling notions in them. Lastly, we also deal with the computability aspect of our set $\mathbb{R}^{\mathbb{Z}_<}$. We define the set $\mathbb{R}_c^{Z_<}$, a set of all computable numbers in $\mathbb{R}^{\mathbb{Z}_<}$, and show that its standard arithmetic operations (functions) are computable. We use a concrete implementation of these ideas in the programming language Python, whose syntax should be intuitively understandable even by those not familiar with it.

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List of symbols

① This denotes the first infinity as introduced by Sergeyev in his Grossone theory.

Δ^m The set of all infinitesimals multiplication of $\boldsymbol{\varepsilon}^m$. More subtle definition of this set can be seen in Definition 4.4.

$\Delta^m_\downarrow$ The set of all infinitesimals multiplication of $\boldsymbol{\varepsilon}^n$ for all $n \geq m$. More subtle definition of this set can be seen in Definition 4.4.

$\mathtt{d}$ This denotes the metric in our set, $\mathbb{R}^{\mathbb{Z}_<}$, as defined in Definition 4.2

$\mathtt{d}_\psi$ This denotes the metric in our set, $\mathbb{R}^{\mathbb{Z}_<}$, as defined in Definition 4.2

ε Together with the other Greek lower-case alphabets, they denote the infinitesimals.

$\widehat{\mathbb{R}}$ A new set of numbers constructed by combining the two set of axioms of $\mathbb{R}$ and ${}^*\mathbb{R}$

${}^*\mathbb{R}$ Set of hyperreal numbers

$\mathbb{Q}$ The set of rational numbers

$\mathbb{R}^{\mathbb{Z}}$ Our new model of numbers which provides the consistency of one chunk of $\widehat{\mathfrak{L}}$

$\mathbb{R}$ Set of real numbes

$\mathbf{x}$ Together with the other bold symbols($\mathbf{y}$, $\mathbf{z}$, $\mathbf{1}$, $\boldsymbol{\varepsilon}$, etc), they denote members of the set $\mathbb{R}^{\mathbb{Z}_<}$

$\widehat{\mathfrak{L}}$ The language of the set $\widehat{\mathbb{R}}$

${}^*\mathfrak{L}$ The language of the set ${}^*\mathbb{R}$

$\mathfrak{L}$ The language of the set $\mathbb{R}$

$\mathrm{CC}(s_n, s)$ This notation denotes that a sequence s_n is classically convergent to s.

$\mathrm{HC}(s_n, s)$ This notation denotes a sequence s_n is hyperconvergent to s.

τ_{St} The standard topology in $\mathbb{R}^{\mathbb{Z}_<}$ which is based on the $\mathtt{St}$-balls in there.

x Together with the other not-bold letter (y, z, a, etc), these letters denote member of classical set of numbers, e.g. $\mathbb{R}$, $\mathbb{N}$, etc.

$\mathrm{RC}(s_n, s)$ This notation denotes that a sequence s_n is $\mathbb{R}^{\mathbb{Z}_<}$-convergent to s.

This research is dedicated especially to all of the people who are searching for the Truth in their lives.

Acknowledgments

It is not a fair task to acknowledge all the people who made this Ph.D. thesis possible with a few words. However, I will try to do my best to extend my great appreciation to “everyone” who helped me through out this research.

I shall begin with God, the Truth *par excellence*: without His will I would have never found the right path. His mercy and love has been throughout my life and ever more during the process of this research.

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"Calvin: If people sat outside and looked at the stars each night, I'll bet they'd live a lot differently.

Hobbes: How so?

Calvin: Well, when you look into infinity, you realize that there are more important things than what people do all day."

Bill Waterson

"Begin at the beginning", the King said gravely, "and go on till you come to the end: then stop."

Lewis Carroll in *Alice in Wonderland*

1

Introduction

1.1 Inconsistencies in Mathematics

There have been many attempts to rule out the existence of inconsistency in mathematical and scientific theories. Since the 1930s, we have known (from Gödel's results) that it is impossible to prove the consistency of any interesting system (in our case, this is a system capable of dealing with arithmetic and analysis). One of the famous examples of inconsistency can be seen in the way of finding the derivative of a function.

Suppose we have a function $f(x) = ax^2 + bx + c$ and want to find its first derivative. By using Newton's 'definition of the derivative':

$$\begin{aligned}
f'(x) &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h} \\
&= \frac{ax^2 + 2axh + 2h^2 + bx + bh + c - ax^2 - bx - c}{h} \\
&= \frac{2axh + 2h^2 + bh}{h} \\
&= 2ax + h + b \qquad (1.1) \\
&= 2ax + b \qquad (1.2)
\end{aligned}$$

In the example above, the inconsistency is located in treating the variable h (some researchers speak of it as an infinitesimal). It is known from the definition that h is a small but non-trivial neighbourhood around x and, because it is used as a divisor, cannot be zero. However, the variable h is simply omitted in the end of the process (from Equations 1.1 to 1.2) indicating that it was, essentially, zero after all. Hence, we have an inconsistency.

This problem of inconsistency has been "resolved" in the 19$^{\text{th}}$ century [1] by the concept of limit, but its (intuitive) naive use

[1] If we look historically, the debates of the use of infinitesimals have a long and vivid history. Their early appearance in mathematics was from the Greek atomist philosopher Democritus (around 450 B.C.E.), only to be dispelled by Eudoxus (a mathematician around 350 B.C.E.) in

is still common nowadays, e.g. in physics. This fact can be seen for example in [43]. In spite of that, interesting and correct results are still obtained. This outlines how firmly inconsistent infinitesimal reasoning (which is a reasoning with *prima facie* inconsistent infinitesimals) is entrenched in our scientific community and it means that inconsistency is something that, if unavoidable, should be handled appropriately. Actually, inconsistency would not have been such a problem if the logic used was not explosive, i.e. we cannot say that from a contradiction, anything can be proved [46]. The problem is that our mathematical theory is mostly based on classical logic, which is explosive. Thus, one promising solution is to change the logic into a non-explosive one and this is the main reason for the birth of paraconsistent mathematics which uses paraconsistent logic as its base.

1.2 Related Works

Doing a mathematical analysis of real structures using paraconsistent logic is a recent and challenging topic. Recent advances have been built on developments in set theory [47], geometry [31], arithmetic [30], and also the elementary research at calculus [30] and [8].

A first thorough study to apply paraconsistent logic in real analysis was based on the early work, such as [10] and [29]. In [28], McKubre-Jordens and Weber analised an axiomatic approach to the real line using paraconsistent logic. They succeed to show that basic field and also compactness theorems hold in that

what was to become "Euclidean" mathematics.

approach. They also could specify where the consistency requirement is necessary. These preliminary works in [28] and [48] show how successful a paraconsistent setting to analysis can be. On the side of non-standard analysis, one field in Mathematics which tries to deal with infinities and infinitesimals, it can be seen for example in [1] and [15] that it is still well-studied and still used in many areas.

1.3 The Underlying Ideas

The underlying ideas of this project are as follows. We have two languages: $\mathfrak{L}$ and $\mathfrak{L}^*$, where the latter language is an extension of the first one. If we want to relate these languages to the two existing mathematical sets, we could say that $\mathfrak{L}$ is the language of real numbers ($\mathbb{R}$) and $^*\mathfrak{L}$ is for hyperreal numbers ($^*\mathbb{R}$). Subsection 2.1.3 gives further explanation about these two languages and it is shown there that $\mathbb{R}$ forms a model for the formulas in $\mathfrak{L}$ and likewise, $^*\mathbb{R}$ forms a model for the formulas in $^*\mathfrak{L}$.

Speaking about the hyperreals, the basic idea of this system is to extend the set $\mathbb{R}$ to include infinitesimal and infinite numbers without changing any of the elementary axioms of algebra. The next interesting (and useful) question is: what properties are preserved in passing between $^*\mathbb{R}$ and $\mathbb{R}$? Here it is the transfer principle which helps to give the answer.

However, there are some problems with the transfer principle, notably its non-computability. Solving this problem is one of the motivations of this research. As we can guess from its name, the transfer principle exists due to the usage of two languages in our theories. Thus, one of the logical ways to solve this problem is by having just one language and so, we try to collapse those two languages into one language $\widehat{\mathfrak{L}}$. The language $\widehat{\mathfrak{L}}$ and its related set, $\widehat{\mathbb{R}}$, can be seen in Chapter 3. Nevertheless, as shown in Examples 2.36 and 3.12, there is at least one big problem from this idea: a contradiction.

We can at present consider two possible ways of resolving this contradiction. The first way is to change the base logic. Note that contradiction might be a problem because the base logic that we use is classical logic where contradiction leads to absurdity. The same problem may not be posed if the base logic is changed to another type of logic which is resilient to contradiction, i.e. paraconsistent logics. And there are many paraconsistent logics that are available at the moment. This could be a good thing, or from another perspective, be an additional difficulty. It is an additional difficulty because, before we immerse ourselves in our main research, we need to choose wisely which paraconsistent logic we want to use, i.e. which one is the most appropriate or the best for our purpose. But then, to be able to choose the "right" one, we need to know first which criteria to use and this in itself is still an open question.

The second way we could consider of resolving the problem of contradiction is to have a subsystem in our theory. This idea arose from a specific reasoning strategy, Chunk and Permeate, which was introduced by Brown and Priest in 2003. See Section 3.2 for further explanation about this strategy. Using this strategy, we will divide our set $\widehat{\mathbb{R}}$ into some consistent chunks and build some permeability relations between them. In our opinion, this second idea makes more sense and promises to be more useful than the first and so the main concern of this research will be aligned with this idea. Moreover, after analising this idea deep enough, we come up with some new interesting and useful notions that will be worth to explore even further.

1.4 Our Goal and Project Significance

The long term goal of this project is to build a new model of the nonstandard analysis[2]. There are at least two possible significant impacts on mathematical research if we can achieve our proposed goal, which are:

1. We would have real numbers, infinities, and infinitesimals in one set and would still be able to do our "usual" analysis in, and with this set.

2. In terms of Gödel's second incompleteness theorem, if we can build a new structure for non-standard mathematical analysis which is resilient to contradiction, we would open the door to having not just a sound, but a complete mathematical theory. To put it simply, like Weber said in [46]:

   > "In light of Gödel's result, an inconsistent foundation for mathematics is the only remaining candidate for completeness."

[2] Nonstandard analysis is a branch of mathematics which introduces hyperreal numbers to allow the existence of infinitesimals.

"It's still magic even if you know how it's done."

Terry Pratchett, *A Hat Full of Sky*

2

Preliminaries

2.1 Reals and Hyperreals

This section gives some important classical backgrounds about the set of real numbers and its extension, the set hyperreal numbers. The material in this section, for the most part, can be found in [18] and [22].

2.1.1 Set of Real Numbers

The Construction of Real Numbers

There are two different ways of approaching real numbers. These are the synthetic (or axiomatic) approach and the explicit construction approach.

The Synthetic (or Axiomatic) Approach One way to define the set pf real numbers is by giving a list of its axioms as a complete ordered field. In this approach, we assume the existence of a number system which consists of a non-empty particular set $\mathbb{R}$, two operators $+$ and $\times$, a binary operator $\leq$ on $\mathbb{R}$ and these axioms:

1. $(\mathbb{R}, +, \times)$ forms a *field*, where *field* is defined as usual.
2. $(\mathbb{R}, \leq)$ forms a totally ordered set[3]:
   (a) Reflexivity: $\forall\, x \in \mathbb{R}, x \leq x$.
   (b) Antisymmetry: $\forall x, y \in \mathbb{R}$ if $x \leq y$ and $y \leq x$ then $x = y$.
   (c) Transitivity: $\forall x, y, z \in \mathbb{R}$ if $x \leq y$ and $y \leq z$ then $x \leq z$.
   (d) Totality: $\forall x, y \in \mathbb{R}$ $x \leq y$ or $y \leq x$.
3. Preservation of order under $+$: $\forall x, y, z \in \mathbb{R}$, if $x \leq y$ then $x + z \leq y + z$.
4. Preservation of order under $\times$: $\forall x, y \in \mathbb{R}$, if $0 \leq x$ then $0 \leq x \times y$.

[3] A totally ordered set needs reflexivity, antisymmetry, transitivity, and totality as its condition. Eliminating the totality requirement will form a **partially ordered set/poset** (or sometimes just called an ordered set).

5. The order $\leq$ is complete: If A is a non-empty subset of $\mathbb{R}$, and if A has an upper bound, then A has a least upper bound l, such that for every upper bound u of A, $l \leq u$. This l is called *supremum* of A, denoted by $\sup\{A\}$[4].

This approach to construct the set of real numbers was also used by McKubre-Jordens and Weber in [28]. Notice that the set of rational numbers, $\mathbb{Q}$, satisfies the first four axioms but not the fifth one.

Proposition 2.1. *Set of rational numbers, $\mathbb{Q}$, is not complete.*

Proof. We want to show that there is a subset of $\mathbb{Q}$ which is bounded above but does not have a supremum in $\mathbb{Q}$. Define a set

$$S = \{x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2\}.$$

Clearly, S is a non-empty subset of $\mathbb{Q}$ and it is bounded above, for example by two. Now suppose that the set $\mathbb{Q}$ is complete and so, S would have a supremum α. If we can show that $\alpha \notin \mathbb{Q}$, then we are done. Assume that $\alpha \in \mathbb{Q}$. By the law of trichotomy, one of the following must be true: $\alpha^2 < 2$, $\alpha^2 > 2$, or $\alpha^2 = 2$. We will show that it is impossible for the first two to be true.

1. First case: $\alpha^2 < 2$.

   With this first case, we have the following inequalities.

[4] This notion of completeness of the order $\leq$ diverges from other kind of non-classical approaches where $\sup\{A\}$ need not exist.

$$
\begin{aligned}
& \alpha^2 && < 2 \\
\Leftrightarrow\ & \alpha^2 + 2\alpha && < 2 + 2\alpha \\
\Leftrightarrow\ & \alpha(\alpha + 2) && < 2(\alpha + 1) \\
\Leftrightarrow\ & \alpha && < \frac{2(\alpha+1)}{\alpha+2} = \beta.
\end{aligned}
$$

However, we also have:

$$
\begin{aligned}
\beta^2 = \left(\frac{2(\alpha+1)}{\alpha+2}\right)^2 &= \frac{4\alpha^2 + 8\alpha + 4}{\alpha^2 + 4\alpha + 4} \\
&= \frac{2\alpha^2 + 8\alpha + 8 + 2(\alpha^2 - 2)}{\alpha^2 + 4\alpha + 4} \\
&< \frac{2(\alpha^2 + 4\alpha + 4)}{\alpha^2 + 4\alpha + 4} = 2
\end{aligned}
$$

which shows that $\beta \in S$. This contradicts our assumption that α is the supremum of S.

2. Second case: $\alpha > \sqrt{2}$.

   Just reverse the inequality sign in the first case.

From the two cases above, we must conclude that $\alpha^2 = 2$ and it is well known that that $\nexists \alpha \in \mathbb{Q}$ such that $\alpha^2 = 2$.

■

The Explicit Construction There are two main ways of constructing real numbers explicitly: Cantor's and Dedekind's construction. The first one is in essence was Cauchy sequences while the latter uses Dedekind cuts. Both of those constructions are described more precisely in Definition 2.7 and Definition 2.11.

Cantor's Construction One well-known real numbers is

$$\pi = 3.14159265358979\ldots.$$

In Cantor's view, number π can be approximated by the sequence of rational numbers:

$$3, \tfrac{31}{10}, \tfrac{314}{100}, \tfrac{3141}{1000}, \tfrac{31415}{10000}, \tfrac{314159}{100000}, \ldots.$$

The main idea behind Cantor's construction of real numbers, beside seeing a real number as a sequence of rational numbers, is using the concept of *Cauchy sequences* (defined in Definition 2.3) and also null sequences (defined in Definition 2.4). But before it, we need the definition of convergence.

Definition 2.2 (Limit in Reals)**.** An infinite sequence $\{x_n\}$ from a normed ordered field S has **limit** b (or we say it **converges to** b) if, given any positive number $k \in S$, there is an integer N such that

$$|x_n - b| < k, \text{ for all } n > N.$$

Note that Definition 2.2 is a classical definition of the limit of a sequence. There may be a distinct notion to define the limit of a sequence paraconsistently.

Definition 2.3 (Cauchy Sequence)**.** The sequence $\{x_n\}$ from a normed ordered field S is a Cauchy sequence if, given any positive number $k \in S$, there is an integer M such that

$$|x_n - x_m| < k, \text{ for all } m, n > M.$$

Note that in a Cauchy sequence, successive terms will get closer together as we go further.

Definition 2.4 (Null Sequence)**.** A null sequence is a sequence which is convergent to zero.

Back to our example of the number π. Besides our first approximation above, this sequence of rational numbers can also approximate π (from its continued fractions expansion [45]):

$$3, \tfrac{22}{7}, \tfrac{333}{106}, \tfrac{355}{113}, \tfrac{52163}{16604}, \tfrac{103993}{33102}, \ldots.$$

The two sequences of rational numbers above are approaching (are converging to) the same number, which is π. The next question then: how do we know if more than one sequences approaches the same real number?

Definition 2.5 (Equivalence of Sequences)**.** Two sequences $x = \{x_n\}$ and $y = \{y_n\}$ are equivalent if the difference sequence, $\{x_n - y_n\}$, is a null sequence.

For having Cantor's construction of real numbers, we need a definition of equivalence classes, naturally defined in the usual way as in Definition 2.6. Cantor's definition of real numbers, $\mathbb{R}$, is given in Definition 2.7.

Definition 2.6 (Equivalence Classes)**.** Given a set X and an equivalence relation $\sim$ on X, the equivalence class of an element a in X is the subset of all elements in X which are equivalent to a, denoted by $[a]$. Notationally, $[a]$ is defined by

$$[a] = \{x \in X | a \sim x\}$$

Definition 2.7 (Reals as Cauchy Sequences)**.** Let S be the set of all Cauchy sequences of rational numbers. The set of real numbers, $\mathbb{R}$, is the set of equivalences classes of S.

DEDEKIND'S CONSTRUCTION In construction of real numbers, a *Dedekind cut*, named after Richard Dedekind, is a partition of the rational numbers into two non-empty sets A and B, such that all elements of A are less than all elements of B, and B has no smallest element. More formally:

Definition 2.8 (Dedekind Cut)**.** A Dedekind cut (or simply a cut) is a pair (A, B)[5] of subset of rational numbers $\mathbb{Q}$ such that:

1. $\mathbb{Q}$ is partitioned by A and B (i.e. $A \cup B = \mathbb{Q}$),
2. neither A nor B is empty,
3. if $x, y \in \mathbb{Q}$, $x < y$ and $y \in A$, then $x \in A$ (A is closed downwards),
4. the set A has no greatest element.

The set B might have a smallest element among the rationals. If B has a smallest element among rational numbers, the cut represents that rational number. Otherwise, that cut represents a unique irrational number which fills the "gap" between A and B.

Example 2.9. This cut represents the irrational number $\sqrt{2}$:

$$A = \{x \in \mathbb{Q} : x^2 < 2 \vee x < 0\}$$

[5]The set B is a complement of A.

and the set B is

$$B = \{y \in \mathbb{Q} : y^2 > 2 \wedge y > 0\}.$$

Definition 2.10 (Negative, Addition, Multiplication , and Inverse of Dedekind Cut)**.** The arithmetic of Dedekind cut is defined as follows.

1. If A determines a Dedekind cut (and so a real number α), then the real number $-\alpha$ is determined by this following set:

$$\{-x \in \mathbb{Q} \mid x \notin A\}.$$

2. If A corresponds to a real number α, and B to a real number β then we define $\alpha + \beta$ to be the Dedekind cut determined by the set:

$$\{x + y \mid x \in A, y \in B\}.$$

3. If A corresponds to a real number α, and B to a real number β then $\alpha \times \beta$ is defined as follows.

   (a) First suppose that $\alpha, \beta \geq 0$. This means that $0 \in A$ or $A = \{x \in \mathbb{Q} \mid x < 0\}$, and similarly for B. Then we define $\alpha \times \beta$ to be the Dedekind cut determined by the set:

$$\{x \times y \mid x \in A, x \geq 0, y \in B, y \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}$$

   (b) Now suppose that $\alpha \geq 0$ and $\beta < 0$. We define

$$\alpha \times \beta = -(\alpha \times (-\beta)).$$

(c) Similarly, if $\alpha < 0$ and $\beta \geq 0$, we define

$$\alpha \times \beta = -((-\alpha) \times \beta),$$

(d) whereas if $\alpha, \beta < 0$, we define

$$\alpha \times \beta = (-\alpha) \times (\beta).$$

4. If A corresponds to a real number α then α^{-1} is defined as follows.

$$\alpha^{-1} = \begin{cases} \{x \mid x < \frac{1}{y}, y \notin A\} & \text{if } \alpha > 0; \\ \{x \mid x < \frac{1}{y}, y \notin A, \text{and } y < 0\} & \text{if } \alpha < 0. \end{cases}$$

Definition 2.11 (Reals as Sets of Dedekind Cut)**.** The set of real numbers, $\mathbb{R}$, is defined as the set of all Dedekind cuts.

Reals as Archimedean Field

One of the special *fields* in mathematics is an Archimedean field. An example of a field is the field of real numbers ($\mathbb{R}$). A field is called an Archimedean field when it satisfies the Archimedean property.

Basically, the Archimedean property is the property of having no infinitely large or infinitely small elements. More formally, an algebraic structure in which any two non-zero elements are comparable, i.e. not one of them is infinitesimal with respect to the other, is called an Archimedean structure. In mathematical notation, an Archimedean structure, A, will hold this condition:

$$\forall x, y > 0 \in A, \exists n \in \mathbb{N} \text{ such that } nx > y.$$

In our case, that algebraic structure is an ordered field.

Theorem 2.12 (Reals as Archimedean Field)**.** $\mathbb{R}$ *is an Archimedean field.*

Proof. Suppose that the theorem is false so that there exists x and y such that $nx \leq y \, \forall n \in \mathbb{N}$. Then y is an upper bound of the set $S = \{nx : n \in \mathbb{N}\}$. By the completeness axiom, S has a supremum and let $a = \sup\{S\}$ so that $nx \leq a$ for all $n \in \mathbb{N}$. Take $n = n + 1$. We have $(n+1)x \leq a$ for all n and so $nx \leq a - x < a$ for all n, i.e. $a - x$ is also an upper bound of S which is smaller than a. This is a contradiction. ■

Notice that the proof above is a proof-by-contradiction, which might give us a problem later on when the logic used is changed from classical logic into paraconsistent logic.

2.1.2 Extension of Reals (Hyperreals)

Definition 2.13 (Infinitesimal and Infinite Number)**.** A number $\varepsilon \neq 0$ in an ordered field is called an infinitesimal if it satisfies:

$$|\varepsilon| < \tfrac{1}{n} \text{ for every ordinary natural number } n = 1, 2, 3, \ldots$$

A number ω in an ordered field is called an infinity if it satisfies:

$$\omega > n \text{ for every ordinary natural number } n = 1, 2, 3, \ldots$$

Theorem 2.12 shows us that infinitesimal (and infinite) numbers cannot exist in $\mathbb{R}$. The formal definition of infinitesimal and infinite numbers can be seen in Definition 2.13. Notice that once we have an inifinitesimal (or infinity), we have many of them. Moreover, all numbers that are multiplies of ε (or ω) are infinitesimals (infinities). See Fig. 2.1.1 for the illustration. The question then: is there any ordered field which has infinitesimals and infinities as its members? Definition 2.14 gives the sense of that kind of field.

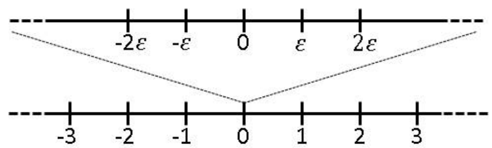


Figure 2.1.1: Infinitesimals around Zero

Definition 2.14 (The set hyperreals as a Field)**.** The set of hyperreal numbers,$^*\mathbb{R}$, is an ordered field that contains the set real numbers as a subfield but also contains infinitesimals and infinities.

Similarly to $\mathbb{R}$, the hyperreals can be built using a procedure similar to Cantor's construction.

Definition 2.15 (Hyperreals in Cantor's View)**.** The set $^*\mathbb{R}$ of hyperreals is the set of all infinite sequences of *real* numbers.

Consider these three following sequences:

$$\mathbf{x} = \{23, 453\pi, -532, 86, 234, -1 - 2\sqrt{5}, \dots\}$$
$$\mathbf{y} = \{2, 2, 2, 2, 2, 2, 2, 2, 2, \dots\}$$
$$\mathbf{z} = \{\pi, 324, -53\sqrt{2}, 11, 346, -1, 2, 2, 2, 2, 2, 2, 2, \dots\}$$

It is clear, based on Definition 2.15, that sequences $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{y}$ are each a hyperreal number. Furthermore, sequences $\mathbf{y}$ and $\mathbf{z}$ actually represent the same hyperreal number. So the question is, when are more than one hyperreals the same?

Definition 2.16 (Equality in Hyperreals)**.** Two hyperreal numbers $\mathbf{x} = \{x_n\}$ and $\mathbf{y} = \{y_n\}$ are equal if $A = \{n \mid x_n = y_n\}$ is a 'big'[6] set of natural numbers. A hyperreal $\mathbf{x} = \{x_n\}$ is positive if $B = \{n | x_n > 0\}$ is a big set of natural numbers.

Arithmetic of Hyperreal Numbers

Members of ${}^*\mathbb{R}$ are called hyperreal numbers, while members of $\mathbb{R}$ are real numbers. This notation can be applied to other sets of numbers, such as hypernaturals for ${}^*\mathbb{N}$, hyperintegers for ${}^*\mathbb{Z}$, hyperrationals for ${}^*\mathbb{Q}$, etc.

A hyperreal number b is:

1. limited iff $\exists r, s \in \mathbb{R}$ such that $r < b < s$.
2. positive unlimited iff $\forall r \in \mathbb{R}, r < b$.
3. negative unlimited iff $\forall r \in \mathbb{R}, b < r$.
4. unlimited[7] iff it is positive or negative unlimited.
5. positive infinitesimal iff $\forall r \in \mathbb{R}^+, 0 < b < r$.
6. negative infinitesimal iff $\forall r \in \mathbb{R}^-, r < b < 0$.
7. infinitesimal iff it is positive or negative infinitesimal.

[6]Note that finite sets are not 'big'. Only an infinite set can be a 'big' set although not all infinite sets are 'big' [18]. Further explanation about 'big' sets can be found also on the same book.

[7]The words "finite" and "infinite" are sometimes interchangeable with "limited" and "unlimited". But actually this does not accord well with the philosophy of our subject.

8. appreciable iff it is limited but not infinitesimal, i.e.

   $\exists r, s \in \mathbb{R}^+ \ni r < |b| < s.$

For the arithmetic of hyperreal numbers, let ε, δ be an infinitesimal, b, c appreciable, and H, K are unlimited. All of the statements in Table 2.1.1 hold [18].

Sums	**Opposites**	**Products**
$\varepsilon + \delta$ is infinitesimal	$-\varepsilon$ is infinitesimal	$\varepsilon.\delta$ and $\varepsilon.b$ are infinitesimal
$b + \varepsilon$ is appreciable	$-b$ is appreciable	$b.c$ is appreciable
$b + c$ is limited (possibly infinitesimal)	$-H$ is unlimited	$b.H$ and $H.K$ are unlimited
$H + \varepsilon$ and $H + b$ is unlimited		
Divisions	**Reciprocals**	**Indeterminate Forms**
$\frac{\varepsilon}{b}$, $\frac{\varepsilon}{H}$ and $\frac{b}{H}$ are infinitesimal	$\frac{1}{\varepsilon}$ is unlimited	$\frac{\varepsilon}{\delta}$, $\frac{H}{K}$, $\varepsilon.H$, $H + K$
$\frac{b}{c}$ is appreciable (if $c \neq 0$)	$\frac{1}{b}$ is appreciable	
$\frac{b}{\varepsilon}$, $\frac{H}{\varepsilon}$ and $\frac{H}{b}$ are unlimited ($\varepsilon, b \neq 0$)	$\frac{1}{H}$ is infinitesimal	

Table 2.1.1: Arithmetic of hyperreal numbers

Halos, Galaxies, and Shadows

How can we compare two hyperreal numbers, b and c? We use the notation of *halo* and *galaxy* as follows:

- Hyperreal b is **infinitely close** to hyperreal c, denoted by $b \simeq c$, if $b - c$ is infinitesimal. Then, the *halo* of b is defined by $\simeq$-equivalence class

  $$\text{hal}(b) = \{c \in {}^*\mathbb{R} | b \simeq c\}$$

- Hyperreal b is **limited distance apart** to hyperreal c, denoted by $b \sim c$, if $b - c$ is limited. Then, the *galaxy* of b is defined by $\sim$-equivalence class

$$\text{gal}(b) = \{c \in {}^*\mathbb{R} \mid b \sim c\}$$

Theorem 2.17 (Arithmetic of Infinite Closeness). *If b, c are limited and $b \simeq b', c \simeq c'$, then $b \pm c \simeq b' \pm c'$, $b.c \simeq b'.c'$, and $\frac{b}{c} \simeq \frac{b'}{c'}$ if $c \not\simeq 0$.*

Proof. We can infer from the hypotheses that:

$$b \simeq b' \text{ so that } b' - b \text{ is infinitesimal and it means that } b' - b \simeq 0, \tag{2.1}$$

and

$$c \simeq c' \text{ so that } c' - c \text{ is infinitesimal and it means that } c' - c \simeq 0. \tag{2.2}$$

1. Because of $(b' \pm c') - (b \pm c) = (b' - b) \pm (c' - c)$ and from Statements 2.1 and 2.2, we get $(b' \pm c') - (b \pm c) \simeq 0$ and it means $b \pm c \simeq b' \pm c'$.

2. Because of $(b'.c') - (b.c) = (b' - b).c' + (c' - c).b$ and from Statements 2.1 and 2.2, we get $(b'.c') - (b.c) \simeq 0$ and it means $b.c \simeq b'.c'$.

3. Because of $\frac{b'}{c'} - \frac{b}{c} = \frac{b'.c - b.c'}{c'.c} = \frac{(b'-b).c - b(c'-c)}{c'.c}$ and from Statements 2.1 and 2.2, we get $\frac{b'}{b} - \frac{b}{c} \simeq 0$ and it means $\frac{b}{c} \simeq \frac{b'}{c'}$. Note that this proof works because of $c \not\simeq 0$. ■

Theorem 2.18 (Shadow). *Every limited hyperreal b is infinitely close to exactly one real number, called the* ***shadow*** *of b, denoted by $\text{sh}(b)$.*[8]

[8]Further down the track, the concept of the "hat" in $\mathbb{R}^{\mathbb{Z}_<}$, which is the standard part of a number in $\mathbb{R}^{\mathbb{Z}_<}$, resembles this notion of shadow in ${}^*\mathbb{R}$.

Proof. Let $A = \{r \in \mathbb{R} : r < b\}$. This set A is not empty as b is limited. By completeness property in $\mathbb{R}$, A has s supremum and let $c \in \mathbb{R}$ be its supremum. We want to show two things: $b \simeq c$ and its uniqueness.

- To show that $b \simeq c$, we show that $b - c$ is infinitesimal, i.e. $|b - c| \leq \varepsilon$. Since c is a supremum, it is also an upper bound of A so that we cannot have $c + \varepsilon \in A$ and it makes $b \leq c + \varepsilon$. Now suppose that $b \leq c - \varepsilon$. But this means that $c - \varepsilon$ would be an upper bound of A (contrary to the fact that c is supremum of A). Hence, $b \nleq c - \varepsilon$, in other words $c - \varepsilon < b$. Altogether, we get $c - \varepsilon < b \leq c + \varepsilon$, so $|b - c| \leq \varepsilon$. Since this holds for any infinitesimal ε, $b \simeq c$.

- For its uniqueness, if $b \simeq c' \in \mathbb{R}$, then as $b \simeq c$, we get $c \simeq c'$ and it follows that $c = c'$ as both are reals. ■

Theorem 2.19 (Arithmatic of Shadows)**.** *If b and c are limited and $n \in \mathbb{N}$, then*

1. $\mathrm{sh}(b \pm c) = \mathrm{sh}(b) \pm \mathrm{sh}(c)$,
2. $\mathrm{sh}(b.c) = \mathrm{sh}(b).\mathrm{sh}(c)$,
3. $\mathrm{sh}(b/c) = \mathrm{sh}(b)/\mathrm{sh}(c)$ *if* $\mathrm{sh}(c) \neq 0$,
4. $\mathrm{sh}(b^n) = \mathrm{sh}(b)^n$,
5. $\mathrm{sh}(|b|) = |\mathrm{sh}(b)|$,
6. $\mathrm{sh}(\sqrt[n]{b}) = \sqrt[n]{\mathrm{sh}(b)}$ *if* $b \geq 0$,
7. *if* $b \leq c$ *then* $\mathrm{sh}(b) \leq \mathrm{sh}(b)$.

2.1.3 The Transfer Principle

Until present, we have been speaking about two sets; one a set of real numbers, the other a set of hyperreal numbers. One of the interesting questions which arises from the study of both is: what properties are preserved in passing between $\mathbb{R}$ and ${}^*\mathbb{R}$? To answer this question, we have a statement called the *transfer principle*. To understand this principle, we should know first about *language* and its model.

Our Formal Language Formal language is built by its syntax and semantics. Here are the symbols that are used in our language:

variables	:	$a\ b\ c \dots\ x_1\ x_2 \dots$
grammatical signs	:	$(\ )\ ,$
connectives	:	$\wedge\ \vee\ \neg\ \rightarrow$
quantifiers	:	$\forall\ \exists$
constant symbols	:	$1\ {-2.5}\ \pi\ \sqrt{5} \dots$
function symbols	:	$+\ -\ \sin\ \tan \dots$
relation symbols	:	$=\ <\ >\ \leq\ \geq \dots$

Syntax gives the rules determining the form that a sentence in a certain language must have in order to be accepted, while semantics gives its meaning (semantics here will be described by a model). Like in natural language, a sentence is built by its term. See Definition 2.20 and 2.22 about term and sentence.

Definition 2.20 (Term). A term is defined recursively by:

1. a constant, or
2. a variable, or
3. $f(t_1, t_2, \ldots, t_n)$ where f is an n-variable function and $t_1, t_2, \ldots, t_n$ are terms.

Example 2.21. 20, π, x, $\sin^{-1}(0.996)$ are terms.

Definition 2.22 (Sentence)**.** A sentence is defined recursively by:

1. $R(t_1, t_2, \ldots, t_n)$ where R is an n-variable relation and $t_1, t_2, \ldots, t_n$ are terms, or
2. $(F \vee G)$ if F and G are already sentences, or
3. $(F \wedge G)$ if F and G are already sentences, or
4. $(F \rightarrow G)$ if F and G are already sentences, or
5. $\forall x\, H(x)$ if x is a variable and $H(x)$ is already a sentence in which x appears in, or
6. $\exists x H(x)$ if x is a variable and $H(x)$ is already a sentence in which x appears in.

Example 2.23. $G(x, y) : x > y$, $(P(x) : x \geq 0) \vee (N(x) : x < 0)$ are sentences.

The example below uses the simple language $\mathfrak{I}$ to build some statements about integer numbers, $\mathbb{Z}$.

Example 2.24. In addition to our usual connectives, variables, quantifiers and grammatical symbols in $\mathbb{Z}$, $\mathfrak{I}$ also contains:

constant symbols	:	$\ldots, -2, -1, 0, 1, 2, \ldots$
function symbols	:	$q(x) = x^2$
		$\text{add}(x, y) = x + y$
		$\text{mul}(x, y) = x \times y$
relation symbols	:	$P(x)$ for "x is positive"
		$E(x, y)$ for "x and y are equal"

In this language $\mathfrak{I}$, one can translate an English statement "squaring any integer number will give a positive number" as $\forall x\, P(s(x))$.

Language for Hyperreals We define a language $\mathfrak{L}$ whose every sentence, if true in reals, is also true in hyperreals.

Definition 2.25 (Language $\mathfrak{L}$)**.** The language $\mathfrak{L}$ consists of the previously listed variables, connectives, and grammatical signs in $\mathbb{R}$, and the following:

constant symbols	:	one symbol for every real number
function symbols	:	one symbol for every real-valued function of any finite number real variables
relation symbols	:	one symbol for every relation on real numbers of any finite number real variables

Like we already said above, semantics in our language will be described by its model. This model gives an interpretation of the sentences of the language such that we may know whether they are true or false in that model.

Definition 2.26 (Model of a Language)**.** Suppose that we have a language $\mathfrak{A}$. A model for $\mathfrak{A}$ consists of:

1. a set A so that each constant symbol in $\mathfrak{A}$ corresponds to an element of A,
2. a set F of functions on A so that each function symbol in $\mathfrak{A}$ corresponds to a function in F,
3. a set R of relations on A so that each relation symbol in $\mathfrak{A}$ corresponds to a relation in R.

Example 2.27. For our language $\mathfrak{I}$ over integer numbers, its model is the set $A = \mathbb{Z}$ with several functions and relations already well-defined in $\mathbb{Z}$. The examples of this function and relation can be seen in Example 2.24.

Theorem 2.28 (Reals as a Model)**.** *The real number system $\mathbb{R}$ is a model for the language $\mathfrak{L}$.*

Proof. Take $A = \mathbb{R}$ and F and R as set of all functions and relations, respectively, which are already well-defined in $\mathbb{R}$. ■

Then, by using the definition of a model, we can define what hyperreal number system is.

Definition 2.29. A hypereal number system is a model for the language $\mathfrak{L}$ that, in addition to all real numbers, contains infinitesimal and infinite numbers.

THE TRANSFER PRINCIPLE Suppose that ${}^*\mathbb{R}$ is the set of all hyperreal numbers. Our goal now is to show that ${}^*\mathbb{R}$ is a model for the language $\mathfrak{L}$. To show this, first, we have to extend the definition of relations and functions on $\mathbb{R}$ into ${}^*\mathbb{R}$.

Definition 2.30 (Extended Relation)**.** Let R be a k-variable relation on $\mathbb{R}$, i.e. for every $x_1, x_2, \ldots, x_k$, $R(x_1, x_2, \ldots, x_k)$ is a sentence that is either true or false. The extension of R to ${}^*\mathbb{R}$ is denoted by ${}^*R$. Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are any hyperreal numbers whose form is $\{x_{1n}\}, \{x_{2n}\}, \ldots, \{x_{kn}\}$, respectively. We define ${}^*R(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_k})$ as **true** iff

$$\{n | R(x_{1n}, x_{2n}, \ldots, x_{kn}) \text{ is true in } \mathbb{R}\}$$

is big. Otherwise, $R(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_k})$ is **false**.

Example 2.31. Suppose that

$$ⓩ = \{1, 2, 3, 4, 5, ...\}$$

By taking $k = 1$ in Definition 2.30, we might have a relation $I(x) =$"x is an integer". The relation $I(ⓩ)$ is true as the set of indexes where relation $I(x)$ is true forms a big set. Because of that, we can conclude that ⓩ is actually a hyperinteger.

Definition 2.32 (Extended Function)**.** Let f be an k-variables function on $\mathbb{R}$. The extension of f to ${}^*\mathbb{R}$ is denoted by ${}^*f$. Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are any hyperreal numbers whose form is $\{x_{1n}\}, \{x_{2n}\}, \ldots, \{x_{kn}\}$, respectively. We define ${}^*f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$ by

$$^*f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) =$$

$$\{f(x_{11}, x_{21}, \ldots, x_{k1}), f(x_{12}, x_{22}, \ldots, x_{k2}), f(x_{13}, x_{23}, \ldots, x_{k3}), \ldots\}$$

Example 2.33. Suppose that

$$ⓔ = \{2, 4, 6, 8, \ldots\}$$

By taking $k = 1$ in Definition 2.32, we might have, for example, a well-defined sinus function. From the definition, we get

$$\sin(ⓔ) = \{\sin(2), \sin(4), \sin(6), \sin(8), \ldots\}$$

Theorem 2.34 (Hyperreals as a Model)**.** *The set of all hyperreal numbers, $^*\mathbb{R}$, forms a hyperreal number system, i.e. $^*\mathbb{R}$ is a model for the language $\mathfrak{L}$ that contains infinitesimals and infinities.*

Proof. Take $A = {^*\mathbb{R}}$ in Definition 2.26 with all of the functions $^*f$ defined in 2.32 and relations $^*R$ defined in 2.30. ■

Definition 2.35 (Transfer Principle)**.** Let S be a sentence in $\mathfrak{L}$. The transfer principle says that:

S is true in the model $\mathbb{R}$ for $\mathfrak{L}$ iff S is true in the model $^*\mathbb{R}$ for $\mathfrak{L}$.

2.1.4 Some Problems

As Goldblatt said in [18], the strength of nonstandard analysis lies in the ability to transfer properties between $\mathbb{R}$ and $^*\mathbb{R}$. But, there are some serious problems with the transfer principle. Some of them are: it is non-computable in the sense of there is no good computable representation of the hyperreals to start with; it really depends intrinsically on the mathematical model or language we use; we are prone to get things wrong when not handled correctly (especially because of human error). One possible solution for overcoming (some of) these problems is **not** to use the transfer principle by combining the language used in reals and the language used in hyperreals. Does this idea pose problems of its own? See the example below.

Example 2.36. Take the well-ordering principle for our example. This principle says that: "every non-empty set of natural numbers contains a least element". Call a set $S = \{x \in N^* : x \text{ is infinite}\}$. Let s be its least element. Note that s is infinite and so $s - 1$. Thus, $s - 1 \in S$ and it makes s is not the least element. Therefore, there exists l such that l is the least element of S and there is no l such that l is the east element of S.

Example 2.36 shows that if we just combine those two languages, it will give us contradictions which can lead to absurdity if the logic used is classical logic. That is why we could use another logic, namely *paraconsistent logic* as it is resilient against local contradiction. And so, the project we have set ourselves in this research is to combine the two languages using a paraconsistent inference strategy, so as to control any contradictions that may arise.

Besides the problems with the transfer principle, there are also some downsides of the construction of the set ${}^*\mathbb{R}$ itself. As we all know, the most common approach to construct a non-standard universe from a standard universe is by taking an ultrapower with respect to some non-principal ultrafilter.

However, as can be seen in [44], this kind of construction requires the axiom of choice whose its validity is still a great deal to discuss. Moreover, it also relies on some heavy and non-constructive mathematical machineries such as Zorn's lemma, the Hahn-Banach theorem, Tychonoff's theorem, the Stone-Cech compactification, or the boolean prime ideal theorem. On the side of the non-standard analysis itself, there are some critiques about it as can be seen in [7, 13, 14, 19, 40, 42]. Most of them are related with its non-constructivism and its difficulties to be used in teaching. This problem can be solved by building a naïve constructive non-standard set and making sure that it is still a useful set by redefining some well-known notions in there. This solution can be seen in Chapters 3 of this work.

2.2 Paraconsistent Logics

Before discussing what paraconsistent logic is, we should know what logic is. Logic is the science of deduction [20]. But, what makes up a logic? Logic is formed by its syntax, semantics, and rules of inference. Syntax designates the symbolic forms that are recognized, semantics are its meaning or its model, while rules of inference are set of rules that allow us to deduct something from some facts or premises in it. Difference in at least one of these three elements will form a different kind of logic.

What then is paraconsistent logic? Generally, paraconsistent logics are logics which permit inference from inconsistent information in a non-trivial fashion [36]. Paraconsistent logics are characterized by rejecting the universal validity of the principle *ex contradictione quodlibet* (ECQ) which is defined below.

Definition 2.37 (ECQ Principle)**.** The principle of explosion, ECQ, is the law which states that any statement can be proven from a contradiction.

What is the consequence of admitting the ECQ principle? It means that if a theory contains a single inconsistency, it becomes absurd or trivial (that is, it has every sentence as a theorem). This is something that, in paraconsistent logics, does not follow necessarily.

Paraconsistent logicians believe that some contradictions does not necessarily make the theory absurd. It just means that one has to be very careful when doing deductions so as to avoid falling from contradiction into an absurdity. In other words, classical and paraconsistent logic treat contradiction in different ways. The former treats contradiction as a global contradiction (making the theory absurd), while the latter treats some contradictions as a local contradiction. In other words, classical logic cannot recognise if there is an interesting structure in the event of a contradiction.

2.2.1 Some Paraconsistent Logics

Formally, paraconsistent logic is defined as in the definition below.

Definition 2.38 (Paraconsistent Logic)**.** Suppose that A is a logical statement. A logic is called paraconsistent logic iff

$$\exists A, B \text{ such that } A \wedge \neg A \nvdash B.$$

The symbol $\Gamma \vdash A$ simply means that there exists a proof of A from set of formulas Γ, in a certain logic.

There are at least two different approaches to paraconsistent logics. The first is by adding another possible value, both true and false, to classical truth values while the second one is called the *relevant-approach*. The idea of the relevant-approach is simply to make sure that the conclusion of an implication must be **relevant** to its premise(s). Here we will discuss two kinds of paraconsistent logics: Priest's Paraconsistent Logic $\mathbf{LP}^{\supset}$[35] and Relevant Logic $\mathbf{R}$[12].[9]

Paraconsistent Logic LP$^{\supset}$

In [34], Priest creates a propositional paraconsistent logic **LP**. The logic $\mathbf{LP}^{\supset}$ is just an extended version of **LP** with an implication connective in it. Some axiom schemes, equivalences and inference rules for $\mathbf{LP}^{\supset}$ can be seen in Table 2.2.2. The formulation in there can also be seen in [2].

The semantics of $\mathbf{LP}^{\supset}$ are obtained using valuations. A valuation for $\mathbf{LP}^{\supset}$ is simply a function v from the set of formulas in $\mathbf{LP}^{\supset}$ to the set $\{t, f, b\}$. This valuation is defined as follows:

[9]Note that in general, relevant logic differs from paraconsistent logic. When someone claims that they use relevant logic, it implies that they use paraconsistent logic, but not vice versa. Using paraconsistent logic does not necessarily mean using relevant logic, e.g. the logic $LP^{\supset}$ below is not relevant logic.

$$v(A \supset B) = \begin{cases} t & \text{if } v(A) = f \\ v(B) & \text{otherwise} \end{cases} \qquad v(\neg A) = \begin{cases} t & \text{if } v(A) = f \\ f & \text{if } v(A) = t \\ b & \text{otherwise} \end{cases}$$

$$v(A \vee B) = \begin{cases} t & \text{if } v(A) = t \text{ or } v(B) = t \\ f & \text{if } v(A) = f \text{ and } v(B) = f \\ b & \text{otherwise} \end{cases}$$

$$v(A \wedge B) = \begin{cases} t & \text{if } v(A) = t \text{ and } v(B) = t \\ f & \text{if } v(A) = f \text{ or } v(B) = f \\ b & \text{otherwise} \end{cases}$$

Axiom Schemes	**Equivalences**
$A \supset (B \supset A)$	$\neg\neg A \equiv A$
$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	$\neg(A \supset B) \equiv A \wedge \neg B$
$((A \supset B) \supset A) \supset A$	$\neg(A \wedge B) \equiv \neg A \vee \neg B$
$(A \wedge B) \supset A$	$\neg(A \vee B) \equiv \neg A \wedge \neg B$
$(A \wedge B) \supset B$	
$A \supset (B \supset (A \wedge B))$	
$A \supset (A \vee B)$	
$B \supset (A \vee B)$	
$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$	
$A \vee \neg A$	

Inference Rule
$\dfrac{A \quad A \supset B}{B}$

Table 2.2.2: Axiom Schemes, Equivalences and Inference Rule in $\mathbf{LP}^{\supset}$

In logical system $\mathbf{LP}^{\supset}$, we have $\Gamma \vdash_{\mathbf{LP}^{\supset}} A$ iff for every valuation v, either $v(B) = f$ for some $B \in \Gamma$ or $v(A) \in \{t, b\}$. Notice that by adding axiom scheme $\neg A \supset (A \supset B)$, we get classical propositional logic. But, if we replace the axiom scheme $A \vee \neg A$ by axiom scheme $\neg A \supset (A \supset B)$ in Table 2.2.2, we will get logic $\mathbf{K}_3^{\supset}$ (Kleene's strong three-valued logic) with an implication connective in it [23].

Relevant Logic R

As stated above, the idea of relevant logic is the relevance between (a) premise(s) and its conclusion, though how to measure the relevance between them is still unclear. Mares in [26] gives axiom schemes and inference rules for logic **R**; see Table 2.2.3. The symbol $\leftrightarrow$ is defined as $(A \supset B) \wedge (B \supset A)$.

We can add two axiom schemes to Table 2.2.3: $A \vee (A \supset B)$ and $A \supset (A \supset A)$ to get another kind of relevant logic: $\mathbf{RM}_3$. It is shown in [11] that by adding *weakening* axiom, i.e. $A \supset (B \supset A)$, to Table 2.2.3, we will get classical propositional logic.

The semantics of logic **R** can be found in [2]. The valuations for logic **R** are similar to the valuations of $\mathbf{LP}^{\supset}$ except for the implication connective which is defined as follow:

$$v(A \supset B) = \begin{cases} t & \text{if } v(A) = f \text{ or } v(B) = t \\ b & \text{if } v(A) = b \text{ and } v(B) = b \\ f & \text{otherwise} \end{cases}$$

Axiom Schemes	
$A \supset A$	$B \supset (A \vee B)$
$(A \supset B) \supset ((B \supset C) \supset (A \supset C))$	$((A \supset B) \wedge (A \supset C) \supset (A \supset (B \wedge C)))$
$A \supset ((A \supset B) \supset B)$	$((A \supset C) \wedge (B \supset C)) \leftrightarrow ((A \vee B) \supset C)$
$(A \supset (A \supset B)) \supset (A \supset B)$	$(A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C)$
$(A \wedge B) \supset A$	$(A \supset \neg B) \supset (B \supset \neg A)$
$(A \wedge B) \supset B$	$\neg\neg A \supset A$
$A \supset (A \vee B)$	

Inference Rule	
$\dfrac{A \quad A \supset B}{B}$	$\dfrac{A \quad B}{A \wedge B}$

Table 2.2.3: Axiom Schemes and Inference Rules in **R**

2.2.2 Paraconsistent Logics in Mathematics

When we are applying paraconsistent logic to a certain theory, there will be at least two terms that we have to be aware of: *inconsistency* and *incoherence*. The first term, inconsistency, is applicable if there occurs a contradiction in a system. Meanwhile, the second term, incoherence, is intended for a system which proves anything (desired or not). In classical logic, there is no difference between these two terms because it admits the ECQ principle. Thus, if a contradiction arises inside a theory, anything that the author would like to say can be proved or inferred within that theory. This is something that likewise does not have to happen n paraconsistent logic.

In mathematical theory, foundation of mathematics is the study of the basic mathematical concepts and how they form more complex structures and concepts. This study is especially important for learning the structures that form the language of mathematics (formulas, theories, definitions, etc.), structures that often called metamathematical concepts. A philosophical dimension is hence central to this study. One of the most interesting topics in the foundation of mathematics is the foundation of real structure, or analysis.

Generally, it is known that the construction of real numbers is categorical in classical logic; while there is advancement in paraconsistent logic [4] though this has not yet been extensively explored. However, it seems viable to make a further study of real structure by developing paraconsistent foundations of analysis.

"A mathematician is a magician who converts adjectives into nouns: continuous into continuum, infinite into infinity, infinitesimal into location..."

Bill Gaede

3

Chunk And Permeate

The transfer principle, which connects the languages of reals ($\mathfrak{L}$) and hyperreals ($\mathfrak{L}^*$), is highly useful for mathematician. As Goldblatt said in [18]: "The strength of non-standard analysis lies in the ability to transfer properties between $\mathbb{R}$ and ${}^*\mathbb{R}$", where $\mathbb{R}$ and ${}^*\mathbb{R}$ are reals and hyperreals, respectively. Nonetheless, there are several serious problems with the transfer principle. Some of them are the fact that it depends intrinsically on the mathematical model or language used, it is prone to error, and its non-computability. One of the ways to avoid the unnecessary complications of the transfer principle is by collapsing the two languages into one language $\widehat{\mathfrak{L}}$.

Simply collapsing these two languages, however, causes additional problems. One of the problems that can be expected to appear is contradiction. However, we can use a paraconsistent logic to handle this if it arises. For being able to speak about this language $\widehat{\mathfrak{L}}$, we need to have a number system on which $\widehat{\mathfrak{L}}$ will be based. We are not necessarily expecting the resulting system to have contradictions, but we will try to make sure we maintain coherence by not allowing contradictions to become an absurdity without further qualification as to *why* they should be avoided.

In this chapter, we will try to construct a number system $\widehat{\mathbb{R}}$ through its axiomatisation. The basic idea of developing this set $\widehat{\mathbb{R}}$ comes from throwing the axioms of $\mathbb{R}$ and ${}^*\mathbb{R}$ together. This set $\widehat{\mathbb{R}}$ will contain positive and negative infinities, and also infinitesimals. It does make sense to insert infinities (and their reciprocals, infinitesimals) into $\widehat{\mathbb{R}}$ as some of the contradictions in mathematics come from their existence and also because they are still used in today's theory as can be seen in [43].

3.1 The Number System $\widehat{\mathbb{R}}$

Axioms 3.1-3.7 give the axiomatisation of our number system $\widehat{\mathbb{R}}$:

Axiom 3.1 (Additive Property of $\widehat{\mathbb{R}}$)**.** In the set $\widehat{\mathbb{R}}$, there is an operator $+$ that satisfies:

A1: For any $x, y \in \widehat{\mathbb{R}}$, $x + y \in \widehat{\mathbb{R}}$.

A2: For any $x, y \in \widehat{\mathbb{R}}$, $x + y = y + x$.

A3: For any $x, y, z \in \widehat{\mathbb{R}}$, $(x + y) + z = x + (y + z)$.

A4: There is $0 \in \widehat{\mathbb{R}}$ such that $x + 0 = x$ for all $x \in \widehat{\mathbb{R}}$.

A5: For each $x \in \widehat{\mathbb{R}}$, there is $-x \in \widehat{\mathbb{R}}$ such that $x + (-x) = 0$.

Axiom 3.2 (Multiplicative Property of $\widehat{\mathbb{R}}$)**.** In the set $\widehat{\mathbb{R}}$, there is an operator $\cdot$ that satisfies:

M1: For any $x, y \in \widehat{\mathbb{R}}$, $x \cdot y \in \widehat{\mathbb{R}}$.

M2: For any $x, y \in \widehat{\mathbb{R}}$, $x \cdot y = y \cdot x$.

M3: For any $x, y, z \in \widehat{\mathbb{R}}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

M4: There is $1 \in \widehat{\mathbb{R}}$ such that $(1 = 0) \to \perp$ and $\forall x \in \widehat{\mathbb{R}}\ x \cdot 1 = x$.

M5: For each $x \in \widehat{\mathbb{R}}$, if $(x = 0) \to \perp$, then there is $y \in \widehat{\mathbb{R}}$ such that $x \cdot y = 1$.

Axiom 3.3 (Distributive Property of $\widehat{\mathbb{R}}$)**.** For all $x, y, z \in \widehat{\mathbb{R}}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Axiom 3.4 (Total Partial Order Property of $\widehat{\mathbb{R}}$)**.** There is a relation $\leq$ in $\widehat{\mathbb{R}}$, such that for each $x, y, z \in \widehat{\mathbb{R}}$:

O1: Reflexivity: $x \leq x$,

O2: Transitivity: $(x \leq y \wedge y \leq z) \to x \leq z$,

O3: Antisymmetry: $(x \leq y \wedge y \leq x) \leftrightarrow x = y$,

O4: Totality: $(x \leq y \to \perp) \to y \leq x$.

O5: Addition order: $x \leq y \to x + z \leq y + z$.

O6: Multiplication order: $(x \leq y \wedge z \geq 0) \to xz \leq yz$.

Axiom 3.5 (Completeness Property of $\widehat{\mathbb{R}}$)**.** Every non-empty bounded above subset of $\widehat{\mathbb{R}}$ has a least upper bound (see Definition 3.8).

Axiom 3.6 (Infinitesimal Property of $\widehat{\mathbb{R}}$)**.** The set $\widehat{\mathbb{R}}$ has an infinitesimal (see Definition 3.9 for what infinitesimal is).

Axiom 3.7 (Archimedean Property of $\widehat{\mathbb{R}}$)**.** For all $x, y > 0$, $\exists n$ such that $nx > y$ (see Definition 3.10 for the operator $>$).

There are several points which should be noted at the outset. The first point is the insistence of using the $(A = B) \to \perp$ notation such as in M4 and M5 in Axiom 3.2. Take the one in M4, for example, that is $(1 = 0) \to \perp$. The reason why we insist of using this notation due to the possible use of paraconsistent logic in our theory. In classical logic, for any statements A and A, $A \neq B$ is equivalent to $(A = B) \to \perp$. However, this is not the case in paraconsistent logic. When we write $A \neq B$, it is still possible that A is actually the same with B even though they might be distinct. If what we intend to say is that it is impossible that $A = B$, then we should have used $(A = B) \to \perp$ which in other words saying that $A = B$ leads to absurdity.

The second point is what Axiom 3.5 and Axioms 3.6 and 3.7 cause. The first axiom, which states the completeness property of $\widehat{\mathbb{R}}$, causes computability issues in our set. The last two axioms, they cause the consistency trouble.

Here are some definitions needed in the axiomatisation above:

Definition 3.8. Suppose that S is a bounded above non-empty subset of $\widehat{\mathbb{R}}$. Classically, an element $t \in \widehat{\mathbb{R}}$ is a least upper bound of S, called $\sup S$ (the supremum of S) iff t is an upper bound of S i.e. $\forall s \in S\ s \leq t$, and $t \leq u$ for every upper bound u of S.

Definition 3.9. An element $x \in \widehat{\mathbb{R}}$ is

- infinitesimal iff $\forall n \in \mathbb{N}\ |x| < \frac{1}{n}$;
- finite iff $\exists r \in \mathbb{R}\ |x| < r$;
- infinite iff $\forall r \in \mathbb{R}\ |x| > r$;
- appreciable iff x is finite but not an infinitesimal;

Definition 3.10. For any numbers $x, y \in \widehat{\mathbb{R}}$,

1. $x \geq y := x < y \rightarrow \perp$
2. $x < y := x \leq y \wedge (x = y \rightarrow \perp)$
3. $x > y := x \geq y \wedge (x = y \rightarrow \perp)$

Definition 3.11. By using notation ε as an infinitesimal, an infinity ω is defined as a reciprocal of ε, i.e. $\frac{1}{\varepsilon}$.

As in Table 2.1.1, for the arithmetic of hyperreal numbers, let $\varepsilon_1, \varepsilon_2$ be an infinitesimal, b, c appreciable, and ω_1, ω_2 are infinite. All of the statements in Table 3.1.1 are true.

If we look further, the set $\widehat{\mathbb{R}}$, which is built on the axioms above, is actually an inconsistent set (there is a contradiction in it). Example 3.12 gives one of these contradictions.

Sums	Opposites	Products
$\varepsilon_1 + \varepsilon_2$ is infinitesimal	$-\varepsilon_1$ is infinitesimal	$\varepsilon_1.\varepsilon_2$ and $\varepsilon_1.b$ are infinitesimal
$b + \varepsilon_1$ is appreciable	$-b$ is appreciable	$b.c$ is appreciable
$b + c$ is finite (possibly infinitesimal)	$-\omega_1$ is infinite	$b.\omega_1$ and $\omega_1.\omega_2$ are infinite
$\omega_1 + \varepsilon_1$ and $\omega_1 + b$ is infinite		
Divisions	**Reciprocals**	**Indeterminate Forms**
$\frac{\varepsilon_1}{b}$, $\frac{\varepsilon_1}{\omega_1}$ and $\frac{b}{\omega_1}$ are infinitesimal	$\frac{1}{\varepsilon_1}$ is infinite	$\frac{\varepsilon_1}{\varepsilon_2}$, $\frac{\omega_1}{\omega_2}$, $\varepsilon_1.\omega_1$, $\omega_1 + \omega_2$
$\frac{b}{c}$ is appreciable (if $c \neq 0$)	$\frac{1}{b}$ is appreciable	
$\frac{b}{\varepsilon_1}$, $\frac{\omega_1}{\varepsilon_1}$ and $\frac{\omega_1}{b}$ are infinite ($\varepsilon_1, b \neq 0$)	$\frac{1}{\omega_1}$ is infinitesimal	

Table 3.1.1: Arithmetic of The Members of $\widehat{\mathbb{R}}$

Example 3.12. Suppose that we have a set $S = \{x \in \widehat{\mathbb{R}} : |x| < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$. In other words, the set S consists of all infinitesimals in $\widehat{\mathbb{R}}$. It is easily proven that S is not empty and bounded above. So by Completeness Axiom, S has a least upper bound. Suppose that z is its least upper bound (which also means that z must be an infinitesimal). Because z is an infinitesimal, $2z$ is also an infinitesimal and this means $2z$ is also in $\widehat{\mathbb{R}}$. By using Definition 3.10, $z < 2z$ and so, z is not the least upper bound of S. Now suppose that $\sup S = 2z$. The same argument can be used to show that $2z$ is not supremum of S but $3z$. We can build this same argument infinitely to show that there does not exist s such that $\sup S = s$. Thus, we have $\exists s : \sup S = s$ and $\nexists s : \sup S = s$.

This kind of contradiction forces us to use a non-explosive logic such as paraconsistent logic instead of classical logic to do our reasoning in $\widehat{\mathbb{R}}$. In this research, we choose a particular paraconsistent reasoning strategy, which is *chunk and permeate* [8], to resolve our dilemma.

3.2 Chunk and Permeate

A paraconsistent logic is a logic that is used to reason about inconsistent premises without exploding in the sense that if a contradiction is found, then everything can be inferred. One particular strategy in this logic is Chunk and Permeate which was introduced by Brown and Priest in [8]. The general idea of this approach is that, given a set of inconsistent premises, instead of reasoning about it as a whole set, one should focus on consistent subsets of premises. In this strategy, an inconsistent theory is broken up into chunks and only limited information is allowed to pass from one chunk to another. Hence, there will be a way to allow partial flow (this is the "permeate" part). The formal idea is given as follow.

Let M be a classical language which contains $\vdash$ as a classical consequence relation, and Σ is a set of sentences in there. We define $\Sigma^{\vdash}$ be the *closure* of Σ under $\vdash$ and a *covering* of Σ as a set $\{\Sigma_i : i \in I\}$, such that $\Sigma = \bigcup \Sigma_i$ where all of the Σ_i are **classically consistent** (there is no contradiction within any of them).

Now suppose that $C = \{\Sigma_1, \Sigma_2, \ldots\}$ is a covering on Σ. Then ρ, a *permeability relation* on C, is a map from $I \times I$ to subsets of the sentences in M. If $i_0 \in I$, we will call a structure $\langle C, \rho, i_0 \rangle$ as a C&P structure on Σ.

If $\mathfrak{S} = \langle C, \rho, i_0 \rangle$ is a C&P structure on Σ and ℓ is a sentence in the language M, then

$$\Sigma \vdash_{\mathfrak{S}} \ell \text{ iff } \ell \in \Sigma_{i_0}^{\omega}$$

where $\vdash_{\mathfrak{S}}$ is a C&P consequence relation of Σ w.r.t. $\mathfrak{S}$. The symbol $\Sigma_{i_0}^{\omega}$, a set of sentences, is defined through:

$$\Sigma_i^\omega = \bigcup_{n<\omega} \Sigma_i^n$$

and Σ_i^n is defined recursively on n as follows:

$$\begin{array}{lcl} \Sigma_i^0 & = & \Sigma_i^{\vdash} \\ \Sigma_i^{n+1} & = & \left(\Sigma_i^n \bigcup_{j\in I} \left(\Sigma_j^n \cap \rho(j,i)\right)\right)^{\vdash}. \end{array}$$

In other words, Σ_i^{n+1} consists of what can be inferred from Σ_i^n together with whatever flows into chunk i from the other chunks at level n. Note that if there are two chunks on Σ, the source chunk Σ_S and the target chunk Σ_T, we call the related C&P structure *binary structure*.

We will now show an example of how to apply C&P strategy to one of the problems in infinitesimal calculus: how to find a derivative of a function. This example can also be seen in [8]. Let M be the language of the second-order theory of the real numbers. Suppose that the language includes:

1. the functional abstraction operator λ (this needs to be a λ-function, so that it is not simply a collection of free variables),
2. one symbol δ (intuitively, δx is an infinitesimal part of x), and
3. a functional D such that, if f is a function, so is Df (intuitively, Df is the derivative of f).

Now we will build a C&P binary structure $\langle\{\Sigma_S, \Sigma_T\}, \rho, T\rangle$. The source chunk Σ_S contains the second-order theory of the reals together with the usual axioms for λ and these two further axioms:

S1 $\quad Df = \lambda x(\left(f(x+\delta x) - fx\right)/\delta x)$

S2 $\quad \forall x\, \delta x \neq 0$

The axiom **S1** is just the definition of a derivative, while axiom **S2** states the nature of an infinitesimal in Σ_S. The target chunk Σ_T is the same as the first, except that instead of containing **S1** or **S2**, it contains:

T1 $\quad \forall x\, \delta x = 0$

This **T1** states the nature of an infinitesimal in this chunk. It is clear that $\Sigma = \Sigma_S \cup \Sigma_T$ is inconsistent (**S2** and **T1** contradict each other). Now we need to define the permeability function, ρ, such that $\rho(S, T)$ is the set of equations of the form $Df = g$, where:

1. neither f nor g contains D (this makes sure that we just allow only one derivation at a time),
2. f is a λ-term containing no occurrences of δ (in other words, f does not have an infinitesimal in it),
3. g is of the form $\lambda x(h + p)$, where h contains no occurrences of δ, and p is a polynomial of powers > 0 of δx (this determines when to stop the differentiation).

Example 3.13 below gives the illustration of how to apply C&P strategy to do differentiation.

Example 3.13. We will find the derivative of the function $\lambda x(4x^3)$. First working

within Σ_S, we have:

$$D\lambda x(4x^3) = \lambda x\left({}^{(\lambda x(4x^3)(x+\delta x) - \lambda x(4x^3)x)}/_{\delta x} \right) \tag{3.1}$$

$$= \lambda x\left({}^{(4(x+\delta x)^3 - 4x^3)}/_{\delta x} \right) \tag{3.2}$$

$$= \lambda x\left({}^{12x^2\delta x + 12(\delta x)^2 + 4(\delta x)^3}/_{\delta x} \right) \tag{3.3}$$

$$= \lambda x(12x^2 + 12\delta x + 4(\delta x)^2) \tag{3.4}$$

The transition from Eq. 3.3 to Eq. 3.4 is permitted since $\delta x \neq 0$ in Σ_S. The next step is to permeate this equation into Σ_T. Since $\delta x = 0$ in Σ_T, we have:

$$D\lambda x(4x^3) = \lambda x(12x^2) \tag{3.5}$$

The Equation 3.5 is what we have known as derivative of $4x^3$, which is $12x^2$.

3.3 Chunks in $\widehat{\mathfrak{L}}$ and The Creation of The Set $\mathbb{R}^{\mathbb{Z}_<}$

In our theory, suppose that $\widehat{\mathfrak{L}}$ is the language where the set $\widehat{\mathbb{R}}$ is. Then we have to divide $\widehat{\mathfrak{L}}$ in to some consistent chunks. Naturally, there might be several ways to do it (e.g. one can have an idea to divide the original set into two, three, or even more chunks). Nevertheless, we found out that one particular way to have the chunks, as provided in Subsection 3.3.1, is the most interesting one as it leads to a creation of a new model of numbers. One of the problems that may arise in this process is to prove the consistency of each chunk. The semantic definition of consistency gives us one technique to do it, i.e. to provide a model for each chunk.

3.3.1 Two Chunks: Axioms (3.1-3.4,3.6) & (3.1-3.5,3.7)

Model for The First Chunk

One of the possible – and interesting – chunks is a set of Axioms 3.1-3.4 and Axiom 3.6.

Here we prove the consistency of this chunk and also some corollaries.

It is well-established that the set of hyperreals, $^*\mathbb{R}$, satisfies those axioms [6]. Nevertheless, the construction of hyperreals $^*\mathbb{R}$ depends on highly non-constructive arguments. In particular, it requires an axiom of set theory, the well-ordering principle, which assumes into existence something that cannot be constructed [22].

Here we take a look at a simpler set. Remember that our set has to contain not just $\mathbb{R}$, but also infinitesimals and infinities (and combinations of the two). We take $\mathbb{R}^{\mathbb{Z}}$, functions from integers to real numbers, as our base set. The member of $\mathbb{R}^{\mathbb{Z}}$ consists of *standard* and *non-standard* parts. The *standard part* of a certain number just showing its finite element (the real part), while the *non-standard part* is for its infinite or infinitesimal part (see Definition 3.14).

Definition 3.14 (Member of $\mathbb{R}^{\mathbb{Z}}$)**.** A typical member of $\mathbb{R}^{\mathbb{Z}}$ has the form $\mathbf{x} = \langle \varepsilon_{-i}, \widehat{x}, \varepsilon_j \rangle$ where $\widehat{x} \in \mathbb{R}$ and ε_n denotes the sequence of the constant part of infininitesimals if $n > 0$, and infinities if $n < 0$.

Notice that the symbol $\widehat{x}$ in Definition 3.14 signs the standard part of a number in $\mathbb{R}^{\mathbb{Z}}$. Thus, the member of $\mathbb{R}^{\mathbb{Z}}$ can be seen as a sequence of infinite numbers. Example 3.15 gives an overview of how to write a number as a member of $\mathbb{R}^{\mathbb{Z}}$.

Example 3.15 (Numbers in $\mathbb{R}^{\mathbb{Z}}$)**.**

1. The number 1 is written as $\mathbf{1} = \langle \ldots 0, 0, \widehat{1}, 0, 0, \ldots \rangle$.

2. The number ε is written as $\boldsymbol{\varepsilon} = \langle \ldots 0, \widehat{0}, 1, 0, \ldots \rangle$.

3. The number ω (one of the infinities) is written as $\boldsymbol{\omega} = \langle \ldots 0, 1, \widehat{0}, 0, \ldots \rangle$.

4. The number $2 + 2\varepsilon - \omega^2$ is written as

   $2 + 2\boldsymbol{\varepsilon} - \boldsymbol{\omega}^2 = \langle \ldots, 0, -1, 0, \widehat{2}, 2, 0, \ldots \rangle$.

By using this kind of form, all of the possible numbers can be written in $\mathbb{R}^{\mathbb{Z}}$. However, this infinite form is problematic in a number of ways. For example, multiplication cannot easily be defined and there might exist multiple inverses if the set $\mathbb{R}^{\mathbb{Z}}$ was going to be used (see Example 3.16).

Example 3.16 (Multiple inverses in $\mathbb{R}^{\mathbb{Z}}$)**.** Suppose that we we want to find the inverse of $\boldsymbol{\omega} \widehat{+} \boldsymbol{\varepsilon}$, i.e. a number $\mathbf{y}$ such that:

$$\mathbf{y} \widehat{\times} (\boldsymbol{\omega} \widehat{+} \boldsymbol{\varepsilon}) = \mathbf{1}. \tag{3.6}$$

By rewriting $\boldsymbol{\omega}$ as $\frac{1}{\boldsymbol{\varepsilon}}$ and from Proposition 3.23,

$$\mathbf{y} \widehat{\times} (\frac{\mathbf{1}}{\boldsymbol{\varepsilon}} \widehat{+} \boldsymbol{\varepsilon}) = (\mathbf{y} \widehat{\times} \frac{\mathbf{1}}{\boldsymbol{\varepsilon}}) \widehat{+} (\mathbf{y} \widehat{\times} \boldsymbol{\varepsilon}).$$

Now suppose that

$$\mathbf{y} = \langle \ldots, y_{-i}, y_{-i+1} \ldots, y_{-2}, y_{-1}, \widehat{y_0}, y_1, y_2, \ldots \rangle \in \mathbb{R}^{\mathbb{Z}_<}.$$

In term of indexing, multiplying $\mathbf{y}$ with $\frac{1}{\boldsymbol{\varepsilon}}$ will shift the sequence one position to the left – that is:

$$\mathbf{y}\widehat{\times}\frac{\mathbf{1}}{\boldsymbol{\varepsilon}} = \langle \ldots, y_{-i+1}, y_{-i+2}, \ldots, y_{-1}, y_0, \widehat{y_1}, y_2, \ldots \rangle \tag{3.7}$$

and multiplying $\mathbf{y}$ with $\boldsymbol{\varepsilon}$ will shift the sequence one position to the right, i.e:

$$\mathbf{y}\widehat{\times}\boldsymbol{\varepsilon} = \langle \ldots, y_{-i-1}, y_{-i}, \ldots, y_{-2}, \widehat{y_{-1}}, y_0, y_1, \ldots \rangle. \tag{3.8}$$

By using Definition 3.18, adding Eqs. 3.7 and 3.8, then apply the result to Eq. 3.6 gives

$$\begin{aligned}
(\mathbf{y}\widehat{\times}\boldsymbol{\varepsilon})\widehat{+}(\mathbf{y}\widehat{\times}\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}) &= \langle \ldots, y_{-i+1} + y_{-i-1}, \ldots, y_{-1} + y_{-3}, y_0 + y_{-2}, \\
&\qquad \widehat{y_1 + y_{-1}}, y_2 + y_0, \ldots \rangle && (3.9) \\
&= \mathbf{1} \\
&= \langle \ldots, 0, 0, \widehat{1}, 0, 0, \ldots \rangle && (3.10)
\end{aligned}$$

Expanding Eqs. 3.9 and 3.10 will produce the following system of equations:

$$
\begin{aligned}
y_1 + y_{-1} &= 1 \\
y_2 + y_0 &= 0 \\
y_3 + y_1 &= 0 \\
y_4 + y_2 &= 0 \\
y_5 + y_3 &= 0 \\
&\vdots \\
y_0 + y_{-2} &= 0 \\
y_{-1} + y_{-3} &= 0 \\
y_{-2} + y_{-4} &= 0 \\
y_{-3} + y_{-5} &= 0 \\
&\vdots \\
y_{-i+1} + y_{-i-1} &= 0 \\
y_{-i} + y_{-i-2} &= 0 \\
&\vdots
\end{aligned}
\tag{3.11}
$$

Now we have to solve the system of equations above. However, if we look carefully, we have an infinite number of free variables which will provide us with an infinite number of solutions, instead of a single solution. Two of those solutions are:

1. Setting $y_1 = 1$ and $y_0 = 0$. By doing this, we will have

$y_{-1} = y_{-3} = y_{-5} = y_{-7} = \cdots = 0, y_3 = y_5 = y_7 = \cdots = -1,$

$y_5 = y_9 = y_13 = \cdots = 1, y_2 = y_4 = y_6 = y_8 = \cdots = 0$, and

$y_{-2} = y_{-4} = y_{-6} = \cdots = 0$. This gives us

$$\mathbf{y} = \langle \ldots, 0, 0, \ldots, 0, \widehat{0}, 1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots \rangle$$

as an inverse of $\boldsymbol{\omega} \widehat{+} \boldsymbol{\varepsilon}$.

2. Setting $y_1 = 0$ and $y_0 = 2$. By doing this, we will have

$y_{-1} = y_{-3} = y_{-5} = y_{-7} = \cdots = 0, y_3 = y_5 = y_7 = \cdots = -1,$

$y_5 = y_9 = y_13 = \cdots = 1, y_2 = -2, y_4 = y_6 = y_8 = \cdots = 2,$

$y_{-2} = -2$,and $y_{-4} = y_{-6} = y_{-8} = \cdots = 0$. This gives us

$$\mathbf{y} = \langle \ldots, 2, 0, 2, 0, 2, 0, -2, 0, \widehat{2}, 1, -2, -1, 2, 1, 2, -1, 2, 1, 2, \ldots \rangle$$

as an inverse of $\boldsymbol{\omega} \widehat{+} \boldsymbol{\varepsilon}$.

As being told before, Example 3.16 demonstrates that having infinite indices both sides will rise the possibility of having multiple inverses. Because of this, the semi-infinite form is motivated and the modified set is denoted by $\mathbb{R}^{\mathbb{Z}_<}$. The only difference between $\mathbb{R}^{\mathbb{Z}_<}$ and $\mathbb{R}$ is that, for any number $\mathbf{x}$, we will not have an infinite sequence on the left side of its standard part.

Example 3.17 (Numbers in $\mathbb{R}^{\mathbb{Z}_<}$)**.**

1. A number 1 is written as $\mathbf{1} = \langle \widehat{1}, 0, 0, \ldots \rangle$.

2. A number ε is written as $\boldsymbol{\varepsilon} = \langle \widehat{0}, 1, 0, \ldots \rangle$.

3. A number ω (one of the infinities) is written as $\boldsymbol{\omega} = \langle 1, \widehat{0}, 0, \ldots \rangle$.

4. A number $2 + 2\varepsilon - \omega^2$ is written as $2 + 2\boldsymbol{\varepsilon} - \boldsymbol{\omega}^2 = \langle -1, 0, \widehat{2}, 2, 0, \cdots \rangle$.

Definition 3.18 (Addition and Multiplication in $\mathbb{R}^{\mathbb{Z}_<}$)**.** For any number $\mathbf{x} = \langle x_z \rangle$ and $\mathbf{y} = \langle y_z \rangle$ in $\mathbb{R}^{\mathbb{Z}_<}$, define:

$$\mathbf{x} + \mathbf{y} = \langle x_z + y_z : z \in \mathbb{Z} \rangle$$

and $\mathbf{x} \times \mathbf{y}$ is calculated by:

$$\mathbf{x} \times \mathbf{y} = \left(\sum_{i=-m} a_i \varepsilon^i \right) \widehat{\times} \left(\sum_{j=-n} b_j \varepsilon^j \right) = \left(\sum_{k \in \mathbb{Z}} c_k \varepsilon^k \right),$$

where $c_k = \sum_{i+j=k} a_i b_j$.

Example 3.19 (Calculating Addition and Multiplication in $\mathbb{R}^{\mathbb{Z}_<}$)**.** Suppose that we have three numbers: $1 + \varepsilon$, $1 + \omega + \omega^2$, and $1 - \omega$. In $\mathbb{R}^{\mathbb{Z}_<}$, those three numbers are written as

$$1 + \boldsymbol{\varepsilon} = \langle \widehat{1}, 1, 0, \cdots \rangle, \boldsymbol{\omega} + \boldsymbol{\omega}^2 + 1 = \langle 1, 1, \widehat{1}, 0, \cdots \rangle, \text{ and}$$
$$-\boldsymbol{\omega} + 1 = \langle -1, \widehat{1}, 0 \cdots \rangle.$$

Then,

1. using Definition 3.18, we have $(1 + \varepsilon) + (1 + \omega + \omega^2) = \langle \widehat{1}, 1, 0, \cdots \rangle \widehat{+} \langle 1, 1, \widehat{1}, 0, \cdots \rangle = \langle 1, 1, \widehat{2}, 1, 0, \cdots \rangle = \omega^2 + \omega + 2 + \varepsilon$ from the following process.

$$
\begin{array}{rcccccc}
 & & & \widehat{1} & 1 & 0 & \cdots \\
\widehat{+} & 1 & 1 & \widehat{1} & 0 & \cdots & \\
\hline
 & 1 & 1 & \widehat{2} & 1 & 0 & \cdots
\end{array}
$$

2. using Definition 3.18, we have

   $(1+\varepsilon) \times (1+\omega+\omega^2) = \langle \widehat{1}, 1, 0, \cdots \rangle \widehat{\times} \langle 1, 1, \widehat{1}, 0, \cdots \rangle =$

   $\langle 1, 2, \widehat{2}, 1, 0, \cdots \rangle = 2+\varepsilon+2\omega+\omega^2$ from the following process.

$$
\begin{array}{rcccc}
 & & 1 & 1 & 1 \\
\times & & & 1 & 1 \\
\hline
 & & 1 & 1 & 1 \\
+ & 1 & 1 & 1 & \\
\hline
 & 1 & 2 & 2 & 1
\end{array}
$$

3. using Definition 3.18, we have $(1+\varepsilon) \times (-\omega+1) =$

   $\langle \widehat{1}, 1, 0, \cdots \rangle \widehat{\times} \langle -1, \widehat{1}, 0, \cdots \rangle = \langle -1, \widehat{0}, 1, 0, \cdots \rangle = -\omega^2 + \varepsilon$ from the following process.

$$
\begin{array}{rccc}
 & & 1 & 1 \\
\times & & -1 & 1 \\
\hline
 & & 1 & 1 \\
+ & -1 & -1 & \\
\hline
 & -1 & 0 & 1
\end{array}
$$

Definition 3.20 (Order in $\mathbb{R}^{\mathbb{Z}_<}$). The set $\mathbb{R}^{\mathbb{Z}_<}$ is endowed with $\widehat{\leq}$, the lexicographical ordering.

Proposition 3.21. *The set $\mathbb{R}^{\mathbb{Z}_<}$ satisfies the additive property in Axiom 3.1.*

PROOF:

1. By Definition 3.18 and because $\mathbb{R}$ is closed under addition, the set $\mathbb{R}^{\mathbb{Z}_<}$ is also closed under $\widehat{+}$ operator. Commutative and associative properties hold also.

2. Define $\mathbf{0} = \langle \widehat{0} \rangle \in \mathbb{R}^{\mathbb{Z}_<}$. By Definition 3.18, for all $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$, $\mathbf{x} \hat{+} \mathbf{0} = \mathbf{x}$.

3. Define $-\mathbf{x} = \langle -x_i, \ldots, -x_{-1}, \widehat{-x_0}, -x_1, \ldots, -x_j \rangle \in \mathbb{R}^{\mathbb{Z}_<}$. By Definition 3.18, for all $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$, there exists $-\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$ such that $\mathbf{x} \widehat{+} (-\mathbf{x}) = \mathbf{0}$.

■

Proposition 3.22. *The set $\mathbb{R}^{\mathbb{Z}_<}$ satisfies the multiplicative property in Axiom 3.2.*

PROOF:

1. By Definition 3.18 and because $\mathbb{R}$ is closed under multiplication, the set $\mathbb{R}^{\mathbb{Z}_<}$ is also closed under $\widehat{\times}$ operator. Similar reason for proving commutative and associative properties.

2. Define $\mathbf{1} = \langle \widehat{1} \rangle$. It is clear that if $\mathbf{1} = \mathbf{0}$ then $1 = 0$, which brings an absurdity. Also, from Definition 3.18, for any $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$,
$\mathbf{x} \widehat{\times} \mathbf{1} = \mathbf{x} \widehat{\times} \langle \widehat{1} \rangle = \mathbf{x}$.

■

Proposition 3.23. *The set* $\mathbb{R}^{\mathbb{Z}_<}$ *satisfies the distributive property in Axiom 3.3.*

PROOF: By considering a member of $\mathbb{R}^{\mathbb{Z}_<}$ as a Laurent series, one can prove that $\mathbf{x}\widehat{\times}(\mathbf{y}\widehat{+}\mathbf{z}) = (\mathbf{x}\widehat{\times}\mathbf{y})\widehat{+}(\mathbf{x}\widehat{\times}\mathbf{z})$. See [24] for its standard proof. ■

Proposition 3.24. *The set* $\mathbb{R}^{\mathbb{Z}_<}$ *satisfies the total order property in Axiom 3.4.*

These following results show how to find an inverse of any members of $\mathbb{R}^{\mathbb{Z}_<}$ and its uniqueness property.

Proposition 3.25. *The number* $1 + \boldsymbol{\omega}$ *has a unique inverse.*

PROOF: We want to find the inverse of $1 + \boldsymbol{\omega}$, i.e. a number $\mathbf{y}$ such that:

$$\mathbf{y}\widehat{\times}(1 + \boldsymbol{\omega}) = \mathbf{1}.$$

Following the similar process in Example 3.16, we will get this system of equations:

$$
\begin{aligned}
y_0 + y_1 &= 1 \\
y_1 + y_2 &= 0 \\
y_2 + y_3 &= 0 \\
y_3 + y_4 &= 0 \\
&\vdots \\
y_{-1} + y_0 &= 0 \\
y_{-2} + y_{-1} &= 0 \\
&\vdots \\
y_{-i+1} + y_{-i+2} &= 0 \\
y_{-i} + y_{-i+1} &= 0
\end{aligned}
\tag{3.12}
$$

Now suppose that $y_0 = x \neq 0$. This particular assignment gives us

$$
\begin{aligned}
y_1 = y_3 = y_5 = y_7 = \cdots &= 1 - x, \\
y_2 = y_4 = y_6 = y_8 = \cdots &= x - 1, \\
y_{-1} = y_{-3} = y_{-5} = y_{-7} = \cdots &= -x \neq 0, \text{ and} \\
y_{-2} = y_{-4} = y_{-6} = y_{-8} = \cdots &= x \neq 0.
\end{aligned}
$$

Because $\mathbf{y}$ must be in $\mathbb{R}^{\mathbb{Z}_<}$ (which means $\exists n < 0$ such that $\forall m < n, y_m = 0$), the last two lines of the equalities above cannot be held and so, y_0 has to be equal to 0.

This final assignment implies that

$$\begin{aligned} y_1 = y_3 = y_5 = y_7 = \cdots = 1, \\ y_2 = y_4 = y_6 = y_8 = \cdots = -1, \text{ and} \\ y_n = 0 \, \forall n < 0. \end{aligned}$$

From the assignments above, we may infer that $\mathbf{y}$, the inverse of $1 + \boldsymbol{\omega}$, is unique and its form is:

$$\mathbf{y} = \langle \widehat{0}, 1, -1, 1, -1, \ldots \rangle = \varepsilon - \varepsilon^2 + \varepsilon^3 - \varepsilon^4 + \ldots$$

■

Proposition 3.26. *The number $\boldsymbol{\omega} \widehat{+} \boldsymbol{\varepsilon}$ has a unique inverse.*

PROOF: We want to find the inverse of $\boldsymbol{\omega} \widehat{+} \boldsymbol{\varepsilon}$, i.e. a number $\mathbf{y}$ such that:

$$\mathbf{y} \widehat{\times} (\boldsymbol{\omega} \widehat{+} \boldsymbol{\varepsilon}) = \mathbf{1}. \tag{3.13}$$

Analogous to the proof of the previous proposition, it can be shown that

$$\mathbf{y} = (\boldsymbol{\varepsilon} \widehat{+} \boldsymbol{\omega})^{-1} = \langle \widehat{0}, 1, 0, -1, 0, 1, 0, -1, 0, \ldots \rangle = \varepsilon - \varepsilon^3 + \varepsilon^5 - \varepsilon^7 + \ldots.$$

■

Lemma 3.27. *For any $r, s \in \mathbb{R}$, a number $r\boldsymbol{\varepsilon} \widehat{+} s\boldsymbol{\omega}$ has a unique inverse.*

PROOF: This is a generalisation of Proposition 3.26. ■

Lemma 3.28. *For any $r \in \mathbb{R}$, a number $r \widehat{+} \boldsymbol{\omega}$ (or $r \widehat{+} \boldsymbol{\varepsilon}$) has a unique inverse.*

PROOF: This is a generalisation of Proposition 3.25. ■

Theorem 3.29. *For any number $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$, $\mathbf{x}$ has a unique inverse.*

PROOF: The proof of this theorem is done by simulating the process in Propositions 3.26 and 3.25 while using an arbitrary number

$$\mathbf{x} = \langle x_{-i}, x_{-i+1}, \ldots, x_{-1}, \widehat{x_0}, x_1, x_2, \ldots \rangle$$

and insisting on left-finiteness. ■

Model for The Second Chunk

The most evident model for this second chunk is the set of real numbers, $\mathbb{R}$. Thus, so far, we have already had two chunks in $\widehat{\mathfrak{L}}$ and we have proved that they are consistent by providing a model for each of them.

3.4 Grossone and The Set $\mathbb{R}^{\mathbb{Z}_<}$

Theories that contain infinities have always been an issue and have attracted much research. Some results can be found in [9, 17, 21, 27, 37]. Note that the arithmetic developed for infinite numbers was quite different with respect to the finite arithmetic that we are used to dealing with. For example, Sergeyev created the grossone theory ten years ago as can be found in [38]. The basic idea of this theory is to treat infinities as usual numbers, so that our usual arithmetic rules apply. He named this infinite number *grossone* and denotes it with ①.

3.4.1 Grossone Theory

There are three three postulates used by Sergeyev in order to build his grossone theory [39]:

1. **Postulate 1**. We postulate the existence of infinite and infinitesimal objects but accept that human beings and machines are able to execute only a finite number of operations.

2. **Postulate 2**. We will not be able to define what the mathematical objects that we deal with are; we will simply construct more powerful tools that will allow us to improve our capacities to observe and to describe the properties of these mathematical objects.

3. **Postulate 3**. We adopt the principle 'the part is less than the whole' for all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).

Grossone is introduced by describing its properties and postulated by the *Infinite Unit Axiom* (IUA) consisting of three parts: Infinity, Identity, and Divisibility. To be more precise, the last axiom is more a definition than an axiom.

Axiom 3.30 (Infinity)**.** For any finite natural number n, it follows that $n < ①$.

Axiom 3.31 (Identity)**.** The following relations link ① to identity elements 0 and 1, in respect to addition and multiplication respectively:

$$0 \cdot ① = ① \cdot 0 = 0, ① - ① = 0$$

$$\frac{①}{①} = 1, ①^0 = 1, 1^{①} = 1.$$

Definition 3.32 (Divisibility)**.** For any finite natural number n, the numbers

$$①, \frac{①}{2}, \frac{①}{3}, \ldots, \frac{①}{n}, \ldots$$

are defined as the number of elements of the nth part of the set $\mathbb{N}$:

$$\begin{gathered} ① = |\{1,2,3,4,5,6,\ldots\}| = |\mathbb{N}| \\ \frac{①}{2} = |\{1,3,5,7,\ldots\}| = |\{2,4,6,8,10,\ldots\}| \\ \frac{①}{3} = |\{1,4,7,10,\ldots\}| = |\{2,5,\ldots\}| = |\{3,6,\ldots\}| \\ \vdots \end{gathered}$$

Obviously, the numeral $\frac{①}{n}$ above is considered to be infinite, and therefore, its inverse $\frac{n}{①}$ is an infinitesimal.

Example 3.33. The set, $\mathbb{E}$, of even natural numbers can be written as

$$\mathbb{E} = \{2, 4, 6, 8, \ldots, ① - 4, ① - 2, ①\}.$$

Thus it follows that the number of elements of the set of even numbers is equal to $\frac{①}{2}$. The other interesting fact is that ① is an even number.

It is very important to emphasise that ① is a number, and so functions as a usual number. For example, there exist numbers such as $① - 100$, $①^3 + 16$, $\ln ①$, and etc. Also for instance, from **Postulate 3**, $① - 1 < ①$. This is unlike the way the usual infinity ∞ behaves where $\infty - 1 = \infty$. It also differs from how Cantor's cardinal numbers behave.

The introduction for this new number ① makes us able to rewrite the set of natural numbers $\mathbb{N}$ as:

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots, ① - 2, ① - 1, ①\}.$$

Furthermore, adding IUA to the axioms of natural numbers will define the set of extended natural numbers ${}^*\mathbb{N}$:

$${}^*\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots, ① - 1, ①\, ① + 1, \ldots, ①^2 - 1, ①^2, \ldots\},$$

and the set ${}^*\mathbb{Z}$, extended integer numbers can be defined from there.

3.4.2 The Model of Grossone Theory

In this subsection, we argue that our new set $\mathbb{R}^{\mathbb{Z}_<}$ provides the model of Grossone theory and therefore prove its consistency rather in a deftly way. Remember that a member of $\mathbb{R}^{\mathbb{Z}_<}$ has the form $\mathbf{x} = \langle \varepsilon_{-i}, \widehat{x}, \varepsilon_j \rangle$ where $\widehat{x} \in \mathbb{R}$ and ε_n denotes the infinite sequence of the constant part of infinitesimals if $n > 0$ and the finite sequence of the constant part of infinities if $n < 0$.

Definition 3.34. In our system $\mathbb{R}^{\mathbb{Z}_<}$, the number ① is written as:

$$① = \langle 1, \widehat{0}, 0, \ldots \rangle.$$

Proposition 3.35. *For every finite number* $\mathbf{r} \in \mathbb{R}^{\mathbb{Z}_<}$, $\mathbf{r} < ①$.

Proof: The order in set $\mathbb{R}^{\mathbb{Z}_<}$ is defined lexicographically. Now suppose that $\mathbf{r} = \langle 0, \widehat{r}, 0, 0, \ldots \rangle$ where $\widehat{r} \in \mathbb{R}$. Then it is clear that $\mathbf{r} < ①$. ■

Proposition 3.36. *All of the equations in Axiom 3.31 are also hold in* $\mathbb{R}^{\mathbb{Z}_<}$.

Proof: Note that in $\mathbb{R}^{\mathbb{Z}_<}$, 0 and 1 are written as $\mathbf{0} = \langle \widehat{0}, 0, \ldots \rangle$ and $\mathbf{1} = \langle \widehat{1}, 0, \ldots \rangle$ respectively. Then by the arithmetic defined in $\mathbb{R}^{\mathbb{Z}_<}$:

1. we have $① - ① = \langle 1, \widehat{0}, 0, \ldots \rangle - \langle 1, \widehat{0}, 0, \ldots \rangle = \langle \widehat{0}, 0, \ldots \rangle = 0$,

2. by treating multiplication like in $\mathbb{R}$ we have $① \cdot 0 = \langle 0, \widehat{0}, 0, \ldots \rangle = 0$ from

$$\begin{array}{ccc} & 1 & 0 \\ \times & & 0 \\ \hline & & 0 \\ + & 0 & \\ \hline & 0 & 0 \end{array}$$

   and $0 \cdot ① = \langle 0, \widehat{0}, 0, \ldots \rangle = 0$ with the same reasoning, and so $① \cdot 0 = 0 \cdot ①$ from since $① \cdot 0 - 0 \cdot ① = 0$ and Lemma **??**.

3. we have $①^0 = \mathbf{1}$ by Definition **??** and by the usual multiplication $\mathbf{1}^{①} = \mathbf{1}$,

4. By definition in our set $\mathbb{R}^{\mathbb{Z}_<}$, $\frac{1}{①} = \frac{1}{\omega} = \varepsilon = \omega^{-1} = ①^{-1}$ and so, $\frac{①}{①} = ① \cdot ①^{-1} = 1$.

■

For the fractional form of ① (Definition 3.32), it can also be defined in $\mathbb{R}^{\mathbb{Z}_<}$ as:

$$\text{for any } n \in \mathbb{N}, \frac{①}{n} = \langle \tfrac{1}{n}, \widehat{0}, 0, \ldots \rangle.$$

Speaking about inverse, one of the advantages of having the set $\mathbb{R}^{\mathbb{Z}_<}$ is to be able to see what the inverse of a number looks like, not like in the Grossone theory. See Example 3.37 for more details.

Example 3.37. In grossone theory, the inverse of $\frac{1}{①} + ①$ is just $\frac{1}{\frac{1}{①}+①}$. While in our set $\mathbb{R}^{\mathbb{Z}<}$, $\frac{1}{①} + ①$ is written as $\varepsilon + \omega$ and its inverse is

$$\langle \widehat{0}, 1, 0, -1, 0, 1, 0, -1, 0, \dots \rangle = \varepsilon - \varepsilon^3 + \varepsilon^5 - \varepsilon^7 + \dots.$$

In other words, the more explicit form of $\frac{1}{\frac{1}{①}+①}$ is a series $(-1)^{n+1}①^{-(2n-1)}$ for $n = 1, 2, 3, \dots \in \mathbb{N}$.

In [25], Gabriele Lolli analysed and built a formal foundation of the grossone theory based on Peano's second order arithmetic. He also gave a slightly different notion of some axioms that Sergeyev used. One of the important theorems in Lolli's paper is the proof that grossone theory – or at least his version of it – is consistent. However, as he said, "The statement of the theorem is of course conditional, as apparent from the proof, upon the consistency of PA^2_μ" while "its model theoretic proof is technically rather demanding".

Thus, through what we have done in this subsection, we have proposed a new way to prove the consistency of grossone theory by providing a *straightforward* model of the theory. There is no need for complicated model-theoretic proofs. The set $\mathbb{R}^{\mathbb{Z}<}$ is enough to establish the consistency of Grossone theory in general. Moreover, the development in the next chapters can also be seen, at least in part, as a contribution to the development of Grossone theory.

Hobbes: "First things first!"

Calvin: "Math will still be there when the snow melts…"

Bill Watterson in *Calvin and Hobbes*

,

4

Topology on $\mathbb{R}^{\mathbb{Z}<}$

In this chapter we discuss some topological properties of the set $\mathbb{R}^{\mathbb{Z}<}$. First, however, there are a number of definitions and issues that should be discussed, in order to understand how they should be applied to our set properly.

4.1 Metrics in $\mathbb{R}^{\mathbb{Z}_<}$

We will define what is meant by a distance (metric) between each pair of elements of $\mathbb{R}^{\mathbb{Z}_<}$.

Definition 4.1. Normally speaking, a *metric* in a set X is a function

$$\rho : X \times X \to [0, \infty)$$

where for all $x, y, z \in X$, these four conditions are satisfied:

1. $\rho(x, y) \geq 0$,
2. $\rho(x, y) = 0$ if and only if $x = y$,
3. $\rho(x, y) = d(y, x)$,
4. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

When that function ρ satisfies all of the four conditions above except the second one, ρ is called a *pseudo-metric* on X.

It is not without reason that we introduce the concept of the pseudo-metric here. This kind of metric will make sense when we are in $\mathbb{R}$. For example, the distance between 0 and ε is 0 as our lens is not strong enough to distinguish those two numbers in $\mathbb{R}$.

Now we define two functions d and d_ψ in $\mathbb{R}^{\mathbb{Z}_<}$ as follows:

Definition 4.2. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}_<}$,

$$\mathtt{d} : \mathbb{R}^{\mathbb{Z}_<} \times \mathbb{R}^{\mathbb{Z}_<} \to \mathbb{R}^{\mathbb{Z}_<} \text{ and } \mathtt{d}_\psi : \mathbb{R}^{\mathbb{Z}_<} \times \mathbb{R}^{\mathbb{Z}_<} \to \mathbb{R}$$

where $\mathtt{d}(\mathbf{x}, \mathbf{y}) = |\mathbf{y} - \mathbf{x}|$ and $\mathtt{d}_\psi(\mathbf{x}, \mathbf{y}) = \mathtt{St}(|\mathbf{y} - \mathbf{x}|)$.

It can be easily verified that $\mathtt{d}$ is a metric in $\mathbb{R}^{\mathbb{Z}_<}$ (and so $\left(\mathbb{R}^{\mathbb{Z}_<}, \mathtt{d}\right)$ forms a metric space) and $\mathtt{d}_\psi$ is a pseudo-metric in $\mathbb{R}^{\mathbb{Z}_<}$ (and so $\left(\mathbb{R}^{\mathbb{Z}_<}, \mathtt{d}_\psi\right)$ forms a pseudo-metric space).

4.2 Balls and Open Sets in $\mathbb{R}^{\mathbb{Z}_<}$

Now that we have the notion of distance in $\mathbb{R}^{\mathbb{Z}_<}$, we can define what it means to be an open set in $\mathbb{R}^{\mathbb{Z}_<}$ by defining first what a ball is in $\mathbb{R}^{\mathbb{Z}_<}$.

Definition 4.3. A ball of radius $\mathbf{y}$ around the point $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$ is

$$B_{\mathbf{x}}(\mathbf{y}) = \{\mathbf{z} \in \mathbb{R}^{\mathbb{Z}_<} \mid d_1(\mathbf{x}, \mathbf{z}) < \mathbf{y}\},$$

where $d_1(\mathbf{x}, \mathbf{y})$ is either $\mathtt{d}(\mathbf{x}, \mathbf{y})$ or $\mathtt{d}_\psi(\mathbf{x}, \mathbf{y})$.

We require this additional definition in order to set forth our explanation about balls properly:

Definition 4.4. The sets Δ^m and $\overset{m}{\Delta^{\downarrow}}$ are defined as follows:

$$\Delta^m = \{\mathbf{x} : \mathbf{x} = a_m \boldsymbol{\varepsilon}^m\} \text{ and } \overset{m}{\Delta^{\downarrow}} = \bigcup_{n \geq m} \Delta^n,$$

where $m \in \mathbb{N} \cup \{0\}$, $a_m \in \mathbb{R}$ and $a_m \neq 0$ whenever $m \geq 1$.

Balls in $\mathbb{R}$ and $\mathbb{R}^{\mathbb{Z}_<}$		
The Set	The Metric	The Form of The Balls
$\mathbb{R}$	$\rho(x,y) = \lvert x-y \rvert$	$B_x(r) = (x-r, x+r)$
$\mathbb{R}^{\mathbb{Z}_<}$	$\mathtt{d}(\mathbf{x},\mathbf{y}) = \lvert \mathbf{x}-\mathbf{y} \rvert$	$B_{\mathbf{x}}(\mathbf{r}) = (\mathbf{x}-\mathbf{r}, \mathbf{x}+\mathbf{r})$
$\mathbb{R}^{\mathbb{Z}_<}$	$\mathtt{d}(\mathbf{x},\mathbf{y}) = \lvert \mathbf{x}-\mathbf{y} \rvert$	$B_{\mathbf{x}}(1/n) = (\mathbf{x}-1/n, \mathbf{x}+1/n)$
$\mathbb{R}^{\mathbb{Z}_<}$	$\mathtt{d}_\psi(\mathbf{x},\mathbf{y}) = \mathtt{St}(\lvert \mathbf{x}-\mathbf{y} \rvert)$	$B_{\mathbf{x}}(1/n) = B_{\mathtt{St}(\mathbf{x})}(1/n) = \{\mathbf{y} \mid \mathtt{St}(\mathbf{y}) \in (\mathtt{St}(\mathbf{x}) - 1/n, \mathtt{St}(\mathbf{x}) + 1/n)\}$

Table 4.2.1: Some types of balls both in $\mathbb{R}$ and $\mathbb{R}^{\mathbb{Z}_<}$

We have to be careful here as unlike in classical topology, there are different notions of balls that can be described as follows. The *first* possible notion of balls is when we use $\mathtt{d}$ as our metric and having $\mathbf{y} > 0$ as our radius. In this case, in $\mathbb{R}^{\mathbb{Z}_<}$, the ball around a point $\mathbf{x}$ with $\mathbf{y}$ radius is just an interval $(\mathbf{x}-\mathbf{y}, \mathbf{x}+\mathbf{y})$. Note that by using $\mathbf{y}$ as the radius, beside having the usual balls with "real" radius (*$\mathtt{St}$-balls*) (that is when $\mathbf{y} = r \in \mathbb{R}$), we also have some infinitesimally small balls (*$\mathfrak{e}$-balls*) when $\mathbf{y} = \mathfrak{e} \in \overset{m}{\Delta}{}^{\downarrow}$ for any given m. The *second* possible notion is while we use the same metric $\mathtt{d}$, we have $1/n$ for some $n \in \mathbb{N}$ as its radius. This will produce balls (*rat-balls*) in the form of $(\mathbf{x}-1/n, \mathbf{x}+1/n)$. The *third* possibility is by using $\mathtt{d}_\psi$ as our metric. In this case, interestingly, the balls around a point $\mathbf{x}$ with $1/n$ radius will be in the form of the following set:

$$\{\mathbf{y} \mid \mathtt{St}(\mathbf{y}) \in (\mathtt{St}(\mathbf{x}) - 1/n, \mathtt{St}(\mathbf{x}) + 1/n)\}.$$

We will call this kind of balls as *ψ-balls*. See Table 4.2.1 for the summary of these possibilities of balls in our sets.

Property 4.5. In $\mathbb{R}$ the $\mathfrak{e}$-ball does not exist, whereas in $\mathbb{R}^{\mathbb{Z}_<}$ there are infinitely many $\mathfrak{e}$-balls around every point there.

Finally, we can define what it means to be an open set in $\mathbb{R}^{\mathbb{Z}_<}$. Notice that because we have two notions of ball in our set – $\mathtt{St}$-balls and $\mathfrak{e}$-ball – it leads us to two different notions of openness.

Definition 4.6 ($\mathtt{St}$-open)**.** A subset $O \subseteq \mathbb{R}^{\mathbb{Z}_<}$ is *$\mathtt{St}$-open* iff

$$\forall x \in O \, \exists n \in \mathbb{N} \text{ s.t. } B_x\left(\tfrac{1}{n}\right) \subseteq O.$$

Definition 4.7 ($\mathfrak{e}$-open)**.** A subset $O \subseteq \mathbb{R}^{\mathbb{Z}_<}$ is *$\mathfrak{e}$-open* iff

$$\forall x \in O \, \exists \mathfrak{e} \in \overset{m}{\Delta^{\downarrow}} \text{ s.t. } B_x(\mathfrak{e}) \subseteq O.$$

Remember that the set $\overset{m}{\Delta^{\downarrow}}$ is defined in Definition 4.4.

Example 4.8. The interval $(\mathbf{2}, \mathbf{3})$ in $\mathbb{R}^{\mathbb{Z}_<}$ is $\mathtt{St}$-open and also $\mathfrak{e}$-open.

Example 4.9. The interval $(\mathbf{0}, \mathfrak{e})$ in $\mathbb{R}^{\mathbb{Z}_<}$ is $\mathfrak{e}$-open, but not $\mathtt{St}$-open.

Example 4.9 gives us the theorem below:

Theorem 4.10. *For any set $U \subseteq \mathbb{R}^{\mathbb{Z}_<}$,*

$$\mathbb{R}^{\mathbb{Z}_<} \models (U \textit{ is } \mathtt{St}\textit{-open} \not\leftrightarrow U \textit{ is } \mathfrak{e}\textit{-open}).$$

Using the two definition of openness above, we can define what it means by two points are topologically distinguishable. There will also be two different notions of distinguishable points as can be seen from Definitions 4.11 and 4.12.

Definition 4.11 ($\mathtt{St}$-distinguishable)**.** Any two points in $\mathbb{R}^{\mathbb{Z}_<}$ are *$\mathtt{St}$-distinguishable* if and only if there is a $\mathtt{St}$-open set containing precisely one of the two points.

Definition 4.12 ($\mathfrak{e}$-distinguishable)**.** Any two points in $\mathbb{R}^{\mathbb{Z}_<}$ are *$\mathfrak{e}$-distinguishable* if and only if there is an $\mathfrak{e}$-open set containing precisely one of the two points.

4.3 Topological Spaces

Definition 4.13. Let X be a non-empty set and τ a collection of subsets of X such that:

T1. $X \in \tau$

T2. $\emptyset \in \tau$

T3. If $O_1, O_2, \ldots, O_n \in \tau$, then $\bigcap_{k=1}^{n} O_k \in \tau$

T4. If $O_a \in \tau$ for all $a \in A$, then $\bigcup_{a \in A} O_a \in \tau$

The pair of objects (X, τ) is called a *topological space* where X is called the *underlying set*, the collection τ is called the *topology* in X, and the members of τ are called *open sets*.

Note that if τ is the collection of open sets of a metric space $(\mathcal{X}, \rho)$, then $(\mathcal{X}, \tau)$ is a *topological metric space*, i.e. a topological space associated with the metric space (X, ρ).

There are at least three interesting topologies in $\mathbb{R}^{\mathbb{Z}_<}$ as can be seen in Definition 4.14 below.

Definition 4.14. The standard topology τ_{St} on the set $\mathbb{R}^{\mathbb{Z}_<}$ is the topology generated by all unions of $\mathtt{St}$-balls. The $\mathfrak{e}$-topology in $\mathbb{R}^{\mathbb{Z}_<}$, $\tau_{\mathfrak{e}}$, is the topology generated by all unions of $\mathfrak{e}$-balls and the third topology in $\mathbb{R}^{\mathbb{Z}_<}$ is pseudo-topology, τ_ψ, when it is induced by $\mathtt{d}_\psi$.

Axiom 4.15. $(\mathbb{R}^{\mathbb{Z}_<}, \tau_n)$, $(\mathbb{R}^{\mathbb{Z}_<}, \tau_\mathfrak{e})$, and $(\mathbb{R}^{\mathbb{Z}_<}, \tau_\psi)$ form topological metric space with $\mathtt{d}$ as their metrics (for the first two) and $\mathtt{d}_\psi$ for the third one.

Theorem 4.16. *$(\mathbb{R}^{\mathbb{Z}_<}, \tau_n)$ is not a Hausdorff space but it is a preregular space.*

Proof: $\mathbb{R}^{\mathbb{Z}_<}$ does not form a Hausdorff space because under the topology τ_{St}, there are two distinct points, $\boldsymbol{\varepsilon} = \langle \widehat{0}, 1\rangle$ and $\boldsymbol{\varepsilon} + \mathbf{1} = \langle \widehat{1}, 1\rangle$ for example, which are not neighbourhood-separable. It is impossible to separate those two points with $\mathtt{St}$-ballss as $1/n > \mathfrak{e}$ for every $n \in \mathbb{N}$ and $\mathfrak{e} \in \overset{m}{\Delta^{\downarrow}}$. However, it is a preregular space as every pair of two $\mathtt{St}$-distinguishable points in $\mathbb{R}^{\mathbb{Z}_<}$ can be separated by two disjoint neighbourhoods. This follows directly from Definition 4.11. ■

Theorem 4.17. *$(\mathbb{R}^{\mathbb{Z}_<}, \tau_\mathfrak{e})$ is a non-connected space and it forms a Hausdorff space.*

Proof: We observe that for all $\mathbf{x}_0 \in \mathbb{R}^{\mathbb{Z}_<}$ and $\mathfrak{e} \in \overset{m}{\Delta^{\downarrow}}$, the balls $B_\mathfrak{e}(\mathbf{x}_0)$ are $\mathfrak{e}$-open and so is the whole space. To show that $\mathbb{R}^{\mathbb{Z}_<}$ is not connected, let

$$S_1 = \{\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<} \mid (\mathbf{x} \leq \mathbf{0}) \text{ or } (\mathbf{x} > \mathbf{0} \text{ and } \mathbf{x} \in \overset{m}{\Delta^{\downarrow}})\} \text{ and}$$
$$S_2 = \{\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<} \mid (\mathbf{x} > \mathbf{0}) \text{ and } \mathbf{x} \notin \overset{m}{\Delta^{\downarrow}}\}.$$

The sets S_1 and S_2 are $\mathfrak{e}$-open, disjoint and moreover, we have that $\mathbb{R}^{\mathbb{Z}_<} = S_1 \cup S_2$ (and so $\mathbb{R}^{\mathbb{Z}_<}$ is not connected). For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}_<}$, $B_x(\mathtt{d}(x,y)/2)$ and $B_y(\mathtt{d}(x,y)/2)$ are open and disjoint. Thus, $\mathbb{R}^{\mathbb{Z}_<}$ forms a Hausdorff space. ■

We will now state the usual definition of the basis of a topology τ.

Definition 4.18. Let (X, τ) be a topological space. A *basis* for the topology τ is a collection $\mathcal{B}$ of subsets from τ such that every $U \in \tau$ is the union of some collections of sets in $\mathcal{B}$, i.e.

$$\forall U \in \tau, \exists \mathcal{B}^* \subseteq \mathcal{B} \text{ s.t. } U = \bigcup_{B \in \mathcal{B}^*} B$$

Example 4.19. On $\mathbb{R}$ with its usual topology, the set $\mathcal{B} = \{(a, b) : a < b\}$ is a topological basis.

Definition 4.20. Let (X, τ) be a topological space and let $x \in X$. A local basis of x is a collection of open neighbourhoods of x, $\mathcal{B}_x$, such that for all $U \in \tau$ with $x \in U$, $\exists B \in \mathcal{B}_x$ such that $x \in B \subset U$.

Definition 4.21. Let (X, τ) be a topological space. Then (X, τ) is first-countable if every point $x \in X$ has a countable local basis.

Definition 4.22. Let (X, τ) be a topological space. Then (X, τ) is second-countable if there exists a basis $\mathcal{B}$ of τ that is countable.

Theorem 4.23. $(\mathbb{R}^{\mathbb{Z}_<}, \tau_{\mathfrak{e}})$ *is first countable but not second-countable.*[10]

Proof: From Axiom 4.15 and because every metric space is first-countable, it follows that $(\mathbb{R}^{\mathbb{Z}_<}, \tau_{\mathfrak{e}})$ is first-countable. However, there cannot be any countable bases in $\tau_{\mathfrak{e}}$ as the uncountably many open sets $O_x = (x - \mathfrak{e}, x + \mathfrak{e})$ are disjoint.

■

[10]Note that the space $(\mathbb{R}^{\mathbb{Z}_<}, \tau_n)$ is still second-countable.

"Wow, it really snowed last night! Isn't it wonderful? Everything familiar has disappeared! The world looks brand new! A new year...a fresh, clean start! It's like having a big white sheet of paper to draw on! A day full of possibilities! It's a magical world, Hobbes, ol' buddy...let's go exploring!"

Bill Watterson in *It's a Magical World: A Calvin and Hobbes Collection*

5

Calculus on $\mathbb{R}^{\mathbb{Z}_<}$

It has been proved previously that the set $\mathbb{R}^{\mathbb{Z}_<}$ forms a field. Remember that for any $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$,

$$\mathbf{x} = \langle x_{-n}, x_{-(n-1)}, \dots, x_{-2}, x_{-1}, \widehat{x}, x_1, x_2, x_3, \dots \rangle$$

where

$$\mathtt{St}(\mathbf{x}) = \widehat{x},$$

$$\texttt{Nst}_\varepsilon(\mathbf{x}) = \{x_1, x_2, x_3, \dots\}, \text{ and}$$
$$\texttt{Nst}_\omega(\mathbf{x}) = \{x_{-n}, x_{-(n-1)}, \dots x_{-1}\}.$$

In other words, for every $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$,

$$\mathbf{x} = \texttt{Nst}_\omega(\mathbf{x}) + \texttt{St}(\mathbf{x}) + \texttt{Nst}_\varepsilon(\mathbf{x}).$$

Note that we can think of $\texttt{St}()$, $\texttt{Nst}_\varepsilon()$, and $\texttt{Nst}_\omega()$ as linear functions – that is for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}_<}$ and a constant $c \in \mathbb{R}$,

$$\texttt{St}(\mathbf{x}+\mathbf{y}) = \texttt{St}(\mathbf{x}) + \texttt{St}(\mathbf{y}), \texttt{St}(c\mathbf{x}) = c\texttt{St}(\mathbf{x}),$$
$$\texttt{Nst}_\varepsilon(\mathbf{x}+\mathbf{y}) = \texttt{Nst}_\varepsilon(\mathbf{x}) + \texttt{Nst}_\varepsilon(\mathbf{y}), \texttt{Nst}_\varepsilon(c\mathbf{x}) = c\texttt{NSt}_\varepsilon(\mathbf{x}),$$
$$\texttt{Nst}_\omega(\mathbf{x}+\mathbf{y}) = \texttt{Nst}_\omega(\mathbf{x}) + \texttt{Nst}_\omega(\mathbf{y}), \text{ and } \texttt{Nst}_\omega(c\mathbf{x}) = c\texttt{Nst}_\omega(\mathbf{x}).$$

Definition 5.1. Suppose that $\texttt{ni}_\varepsilon(\mathbf{x})$ denotes the non-infinitesimal part of $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$, i.e. $\texttt{ni}_\varepsilon(\mathbf{x}) = \texttt{Nst}_\omega(\mathbf{x}) + \texttt{St}(\mathbf{x})$ and function in $\mathbb{R}^{\mathbb{Z}_<}$ be defined in the usual way. Then a function f in $\mathbb{R}^{\mathbb{Z}_<}$ is *microstable* if and only if

$$\texttt{ni}_\varepsilon(f(x+\varepsilon)) = \texttt{ni}_\varepsilon(f(x)),$$

Example 5.2. Suppose that a function f in $\mathbb{R}^{\mathbb{Z}_<}$ is defined as follows:

$$f(\mathbf{x}) = \begin{cases} 1, & \text{if } \texttt{St}(\mathbf{x}) > 0 \\ 0, & \text{else.} \end{cases}$$

Then $f(\mathbf{x})$ is a microstable function.

Theorem 5.3. *Microstability is closed under addition, multiplication, and composition.*

Proof: We want to prove that if f, g are functions defined in $\mathbb{R}^{\mathbb{Z}_<}$ and are microstable, then the functions $f+g, fg, f \circ g$ are also microstable. From Definition 5.1, $\mathtt{ni}_\varepsilon(f(\mathbf{x}+\boldsymbol{\varepsilon})) = \mathtt{ni}_\varepsilon(f(\mathbf{x}))$ and $\mathtt{ni}_{\boldsymbol{\varepsilon}}(g(\mathbf{x}+\boldsymbol{\varepsilon})) = \mathtt{ni}_\varepsilon(g(\mathbf{x}))$. Then:

1. For the function $f+g$:

$$\begin{aligned}
\mathtt{ni}_\varepsilon((f+g)(\mathbf{x}+\boldsymbol{\varepsilon})) &= \mathtt{ni}_{\boldsymbol{\varepsilon}}(f(\mathbf{x}+\boldsymbol{\varepsilon}) + g(\mathbf{x}+\boldsymbol{\varepsilon})) \\
&= \mathtt{ni}_\varepsilon(f(\mathbf{x}+\boldsymbol{\varepsilon})) + \mathtt{ni}_\varepsilon(g(\mathbf{x}+\boldsymbol{\varepsilon})) \\
&= \mathtt{ni}_{\boldsymbol{\varepsilon}}(f(\mathbf{x})) + \mathtt{ni}_{\boldsymbol{\varepsilon}}(g(\mathbf{x})) \\
&= \mathtt{ni}_{\boldsymbol{\varepsilon}}(f(\mathbf{x}) + g(\mathbf{x})) \\
&= \mathtt{ni}_\varepsilon((f+g)(\mathbf{x})).
\end{aligned}$$

2. For the function fg:

$$\begin{aligned}
\mathtt{ni}_\varepsilon(fg(\mathbf{x}+\boldsymbol{\varepsilon})) &= \mathtt{ni}_\varepsilon(f(\mathbf{x}+\boldsymbol{\varepsilon})g(\mathbf{x}+\boldsymbol{\varepsilon}) \\
&= \mathtt{ni}_\varepsilon\big((\mathtt{ni}_{\boldsymbol{\varepsilon}}(f(\mathbf{x}+\boldsymbol{\varepsilon})) + \mathtt{Nst}_\omega(f(\mathbf{x}+\boldsymbol{\varepsilon})))(\mathtt{ni}_{\boldsymbol{\varepsilon}}(g(\mathbf{x}+\boldsymbol{\varepsilon}))+ \\
&\quad \mathtt{Nst}_\omega(g(\mathbf{x}+\boldsymbol{\varepsilon})))\big) \qquad (5.1) \\
&= \mathtt{ni}_\varepsilon(\mathtt{ni}_\varepsilon(f(\mathbf{x}+\boldsymbol{\varepsilon}))\mathtt{ni}_\varepsilon(g(\mathbf{x}+\boldsymbol{\varepsilon}))) \qquad (5.2) \\
&= \mathtt{ni}_\varepsilon(\mathtt{ni}_\varepsilon(f(\mathbf{x}))\mathtt{ni}_\varepsilon(g(\mathbf{x}))) \\
&= \mathtt{ni}_\varepsilon\left((\mathtt{ni}_\varepsilon(f(\mathbf{x})) + \mathtt{Nst}_\omega(f(\mathbf{x})))(\mathtt{ni}_\varepsilon(g(\mathbf{x})) + \mathtt{Nst}_\omega(g(\mathbf{x})))\right) \\
&= \mathtt{ni}_\varepsilon(f(\mathbf{x})g(\mathbf{x})) \\
&= \mathtt{ni}_\varepsilon(fg(\mathbf{x})).
\end{aligned}$$

The movement from Equation (5.1) to Equation (5.2) is allowed because for any function f, $\texttt{Nst}(f)$ can be ignored.

3. For the function $f \circ g$:

$$\begin{aligned}
\texttt{ni}_\varepsilon((f \circ g)(\mathbf{x} + \boldsymbol{\varepsilon})) &= \texttt{ni}_\varepsilon(f(g(\mathbf{x} + \boldsymbol{\varepsilon}))) \\
&= \texttt{ni}_\varepsilon(f(\texttt{ni}_\varepsilon(g(\mathbf{x} + \boldsymbol{\varepsilon})) + \texttt{Nst}_\omega(g(\mathbf{x}+ \\
mathbfepsilon)))) \\
&= \texttt{ni}_\varepsilon(f(\texttt{ni}_\varepsilon(g(\mathbf{x} + \boldsymbol{\varepsilon})))) \\
&= \texttt{ni}_\varepsilon(f(\texttt{ni}_\varepsilon(g(\mathbf{x})))) \\
&= \texttt{ni}_\varepsilon(f(\texttt{ni}_\varepsilon(g(\mathbf{x})) + \texttt{Nst}_\omega(g(\mathbf{x})))) \\
&= \texttt{ni}_\varepsilon(f(g(\mathbf{x}))) \\
&= \texttt{ni}_\varepsilon(f \circ g(\mathbf{x})).
\end{aligned}$$

■

Now for every function f defined in $\mathbb{R}^{\mathbb{Z}_<}$, we are going to have the operator $\texttt{Der}_f$ which takes a 2-tuple in $\left(\mathbb{R} \times \mathbb{R}^{\mathbb{Z}_<}\right)$ as its input and returns a member of $\mathbb{R}^{\mathbb{Z}_<}$ as the output, i.e.:

$$\texttt{Der}_f : \mathbb{R} \times \mathbb{R}^{\mathbb{Z}_<} \to \mathbb{R}^{\mathbb{Z}_<}.$$

Eventually, this operator will be called a *derivative* of f.

Using Newton's original definition (and a slight change of notation), if a function $f(x)$ is differentiable, then its derivative is given by:

$$\texttt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) = \frac{f(\mathbf{x} + \boldsymbol{\varepsilon}) - f(\mathbf{x})}{\boldsymbol{\varepsilon}}. \tag{5.3}$$

Now suppose that we want to find a derivative of f where f is a function defined in $\mathbb{R}^{\mathbb{Z}_<}$. We can certainly use Equation (5.3) to calculate it as that equation holds for any function f. But how is this calculation related to the calculus practised in classical mathematics? Note that using Newton's definition to calculate the derivative will necessarily involve an inconsistent step. This inconsistency is located in the treatment given to the infinitesimal number. Thus it makes sense that in order to explore the problem posed above, we will use a paraconsistent reasoning strategy which is called Chunk and Permeate.

5.1 Chunk and Permeate for Derivative in $\mathbb{R}^{\mathbb{Z}_<}$

Before applying the Chunk & Permeate strategy for derivative in $\mathbb{R}^{\mathbb{Z}_<}$, define a set E which consists of any algebraic terms such that they satisfy:

$$\mathtt{St}(\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})) = f'(x),$$

where $f'(x)$ denotes the usual derivative of f in $\mathbb{R}$. We will need this set E when we try to define the permeability relation between chunks.

Proposition 5.4. *The set E as defined above is inhabited.*

Proof: We want to show that the set E has at least one element in it. It is clear that the identity function $\mathtt{id}(x) = x$ is in E because for all ε:

$$\mathtt{St}(\mathtt{Der}_x(\mathbf{x}, \boldsymbol{\varepsilon})) = \mathtt{St}\left(\frac{\mathbf{x} + \boldsymbol{\varepsilon} - \mathbf{x}}{\boldsymbol{\varepsilon}}\right)$$

$$= \mathtt{St}\left(\frac{\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}}\right)$$

$$= \mathtt{St}(\mathbf{1}) = 1 = f'(x).$$

■

Theorem 5.5. *If f and g are microstable functions in E and c is any real constant, then*

1. *$f \pm g$ are in E,*
2. *cf is in E,*
3. *fg is in E,*
4. *$\frac{f}{g}$ is in E, and*
5. *$f \circ g$ is in E.*

PROOF: Here we calculate the derivative of each of the functions first and will find the standard part afterwards.

1. Using Eq. 5.3 we have

$$\begin{aligned}
\mathtt{Der}_{f+g}(\mathbf{x}, \boldsymbol{\varepsilon}) &= \frac{(f+g)(\mathbf{x}+\boldsymbol{\varepsilon}) - (f+g)(\mathbf{x})}{\boldsymbol{\varepsilon}} \\
&= \frac{f(\mathbf{x}+\boldsymbol{\varepsilon}) + g(\mathbf{x}+\boldsymbol{\varepsilon}) - (f(\mathbf{x}) + g(\mathbf{x}))}{\boldsymbol{\varepsilon}} \\
&= \frac{f(\mathbf{x}+\boldsymbol{\varepsilon}) - f(\mathbf{x}) + g(\mathbf{x}+\boldsymbol{\varepsilon}) - g(\mathbf{x})}{\boldsymbol{\varepsilon}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f(\mathbf{x}+\boldsymbol{\varepsilon}) - f(\mathbf{x})}{\boldsymbol{\varepsilon}} + \frac{g(\mathbf{x}+\boldsymbol{\varepsilon}) - g(\mathbf{x})}{\boldsymbol{\varepsilon}} \\
&= \mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) + \mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}).
\end{aligned}
$$

Then

$\mathtt{St}(\mathtt{Der}_{f+g}(\mathbf{x}, \boldsymbol{\varepsilon})) = \mathtt{St}(\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})) + \mathtt{St}(\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})) = f'(x) + g'(x)$,

and so $f + g$ is in E.

2. Using Eq. 5.3 we have

$$
\begin{aligned}
\mathtt{Der}_{cf}(\mathbf{x}, \boldsymbol{\varepsilon}) &= \frac{cf(\mathbf{x}+\boldsymbol{\varepsilon}) - cf(\mathbf{x})}{\boldsymbol{\varepsilon}} \\
&= \frac{c(f(\mathbf{x}+\boldsymbol{\varepsilon}) - f(\mathbf{x})}{\boldsymbol{\varepsilon}} \\
&= c\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}).
\end{aligned}
$$

Then $\mathtt{St}(\mathtt{Der}_{cf}(\mathbf{x}, \boldsymbol{\varepsilon})) = \mathtt{St}(c\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})) = c\mathtt{St}\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) = cf'(x)$, and so the function cf is in E.

3. Using Eq. 5.3 we have

$$
\begin{aligned}
\mathtt{Der}_{fg}(\mathbf{x}, \boldsymbol{\varepsilon}) &= \frac{(fg)(\mathbf{x}+\boldsymbol{\varepsilon}) - (fg)(\mathbf{x})}{\boldsymbol{\varepsilon}} \\
&= \frac{f(\mathbf{x}+\boldsymbol{\varepsilon})g(\mathbf{x}+\boldsymbol{\varepsilon}) - (f(\mathbf{x})g(\mathbf{x})}{\boldsymbol{\varepsilon}} \\
&= \frac{f(\mathbf{x}+\boldsymbol{\varepsilon})g(\mathbf{x}+\boldsymbol{\varepsilon}) - f(\mathbf{x}+\boldsymbol{\varepsilon})g(\mathbf{x}) + f(\mathbf{x}+\boldsymbol{\varepsilon})g(\mathbf{x}) - f(\mathbf{x})g(\mathbf{x})}{\boldsymbol{\varepsilon}}
\end{aligned}
$$

$$= \frac{f(\mathbf{x}+\boldsymbol{\varepsilon})(g(\mathbf{x}+\boldsymbol{\varepsilon}-g(\mathbf{x}))+g(\mathbf{x})(f(\mathbf{x}+\boldsymbol{\varepsilon})-f(\mathbf{x})}{\boldsymbol{\varepsilon}}$$

$$= f(\mathbf{x}+\boldsymbol{\varepsilon})\left(\frac{g(\mathbf{x}+\boldsymbol{\varepsilon})-g(\mathbf{x})}{\boldsymbol{\varepsilon}}\right)+g(\mathbf{x})\left(\frac{f(\mathbf{x}+\boldsymbol{\varepsilon})-f(\mathbf{x})}{\boldsymbol{\varepsilon}}\right)$$

$$= f(\mathbf{x}+\boldsymbol{\varepsilon})\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon})+g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})$$

Then

$$\begin{aligned}
\mathtt{St}(\mathtt{Der}_{fg}(\mathbf{x},\boldsymbol{\varepsilon})) &= \mathtt{St}(f(\mathbf{x}+\boldsymbol{\varepsilon})\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon})+g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= \mathtt{St}(f(\mathbf{x}+\boldsymbol{\varepsilon})\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{St}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= \mathtt{St}((\mathtt{St}(f(\mathbf{x}+\boldsymbol{\varepsilon}))+\mathtt{Nst}(g(\mathbf{x}+\boldsymbol{\varepsilon}))\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+ \\
&\quad \mathtt{St}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= \mathtt{St}(\mathtt{St}(f(\mathbf{x}+\boldsymbol{\varepsilon}))\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{St}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= \mathtt{St}(\mathtt{St}(f(\mathbf{x}))\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{St}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= \mathtt{St}((\mathtt{St}(f(\mathbf{x}))+\mathtt{Nst}(f(\mathbf{x})))\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+ \\
&\quad \mathtt{St}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= \mathtt{St}(f(\mathbf{x})\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{St}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= \mathtt{St}(f(\mathbf{x})\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{NSt}(f(\mathbf{x})\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+ \\
&\quad \mathtt{St}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{NSt}(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})) \\
&= f(\mathbf{x})\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon})+g(\mathbf{x})\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon}) \\
&= f(\mathbf{x})(\mathtt{St}(\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{Nst}(\mathtt{Der}_g(\mathbf{x},\boldsymbol{\varepsilon})))+ \\
&\quad g(\mathbf{x})(\mathtt{St}(\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon}))+\mathtt{Nst}(\mathtt{Der}_f(\mathbf{x},\boldsymbol{\varepsilon})))
\end{aligned}$$

$$= f(\mathbf{x})(\mathtt{St}(\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon}))) + g(\mathbf{x})(\mathtt{St}(\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})))$$
$$= f(x)g'(x) + g(x)f'(x)$$
$$= (fg)'(x)$$

and so fg is in E.

4. Using Eq. 5.3 we have

$$\begin{aligned}
\mathtt{Der}_{f/g}(x, \boldsymbol{\varepsilon}) &= \frac{(f/g)((\mathbf{x}) + \boldsymbol{\varepsilon}) - (f/g)(\mathbf{x})}{\boldsymbol{\varepsilon}} \\
&= \frac{\dfrac{f((\mathbf{x}) + \boldsymbol{\varepsilon})}{g((\mathbf{x}) + \boldsymbol{\varepsilon})} - \dfrac{f(\mathbf{x})}{g(\mathbf{x})}}{\boldsymbol{\varepsilon}} \\
&= \frac{f((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x}) - f(\mathbf{x})g((\mathbf{x}) + \boldsymbol{\varepsilon})}{\boldsymbol{\varepsilon} g((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x})} \\
&= \frac{f((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x}) - f(\mathbf{x})g(\mathbf{x}) + f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{x})g((\mathbf{x}) + \boldsymbol{\varepsilon})}{\boldsymbol{\varepsilon} g((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x})} \\
&= \frac{1}{g((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x})} \left(\frac{f((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x}) - f(\mathbf{x})g(\mathbf{x}) + f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{x})g((\mathbf{x}) + \boldsymbol{\varepsilon})}{\boldsymbol{\varepsilon}} \right) \\
&= \frac{1}{g((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x})} \left(\frac{f((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x}) - f(\mathbf{x})g(\mathbf{x})}{\boldsymbol{\varepsilon}} + \frac{f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{x})g((\mathbf{x}) + \boldsymbol{\varepsilon})}{\boldsymbol{\varepsilon}} \right) \\
&= \frac{1}{g((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x})} \left(g(\mathbf{x}) \frac{f((\mathbf{x}) + \boldsymbol{\varepsilon}) - f(\mathbf{x})}{\boldsymbol{\varepsilon}} - f(\mathbf{x}) \frac{g((\mathbf{x}) + \boldsymbol{\varepsilon}) - g(\mathbf{x})}{\boldsymbol{\varepsilon}} \right)
\end{aligned}$$

$$= \frac{1}{g((\mathbf{x}) + \boldsymbol{\varepsilon})g(\mathbf{x})} \left(g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})\right)$$

Then

$$
\begin{aligned}
\mathtt{St}(\mathtt{Der}_{f/g}(\mathbf{x}, \boldsymbol{\varepsilon})) &= \mathtt{St}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{g(\mathbf{x} + \boldsymbol{\varepsilon})g(\mathbf{x})}\right) \\
&= \mathtt{St}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{(\mathtt{St}(g(\mathbf{x} + \boldsymbol{\varepsilon})) + \mathtt{Nst}(g(\mathbf{x} + \boldsymbol{\varepsilon})))g(\mathbf{x})}\right) \\
&= \mathtt{St}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{\mathtt{St}(g(\mathbf{x} + \boldsymbol{\varepsilon}))g(\mathbf{x})}\right) \\
&= \mathtt{St}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{\mathtt{St}(g(\mathbf{x}))g(\mathbf{x})}\right) \\
&= \mathtt{St}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{(\mathtt{St}(g(\mathbf{x})) + \mathtt{Nst}(g(\mathbf{x})))g(\mathbf{x})}\right) \\
&= \mathtt{St}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{g(\mathbf{x})g(\mathbf{x})}\right) \\
&= \mathtt{St}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{g(\mathbf{x})g(\mathbf{x})}\right) + \\
&\quad \mathtt{NSt}\left(\frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{g(\mathbf{x})g(\mathbf{x})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{g(\mathbf{x})\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) - f(\mathbf{x})\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})}{g(\mathbf{x})g(\mathbf{x})} \\
&= \frac{\begin{array}{r} g(\mathbf{x})(\mathtt{St}(\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})) + \mathtt{Nst}(\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}))) - f(\mathbf{x}) \\ (\mathtt{St}(\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})) + \mathtt{Nst}(\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon}))) \end{array}}{g(\mathbf{x})g(\mathbf{x})} \\
&= \frac{g(\mathbf{x})(\mathtt{St}(\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}))) - f(\mathbf{x})(\mathtt{St}(\mathtt{Der}_g(\mathbf{x}, \boldsymbol{\varepsilon})))}{g(\mathbf{x})g(\mathbf{x})} \\
&= \frac{g(\mathbf{x})f'(\mathbf{x}) - f(\mathbf{x})g'(\mathbf{x})}{g(\mathbf{x})g(\mathbf{x})} \\
&= \left({}^{f}\!/_{g}\right)'(x)
\end{aligned}
$$

and so f/g is in E. ■

Now we are ready to construct the chunk and permeate structure, called $\widehat{\mathfrak{R}}$, which is formally written as $\widehat{\mathfrak{R}} = \langle \{\Sigma_S, \Sigma_T\}, \rho, T\rangle$ where the source chunk Σ_S is the language of $\mathbb{R}^{\mathbb{Z}_<}$, the target chunk Σ_T is the language of $\mathbb{R}$, and ρ is the permeability relation between S and T.

The source chunk Σ_S As stated before, this chunk is actually the language of the set $\mathbb{R}^{\mathbb{Z}_<}$ and therefore, it consists of all six of its axioms. The source chunk requires one additional axiom to define what it means by derivative. This additional axiom can be stated as:

$$\text{S1: } Df = \mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})$$

where $\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})$ is defined in Equation (5.3).

THE TARGET CHUNK Σ_T Again, the target chunk contains the usual axiom for the set or real numbers, $\mathbb{R}$. There is only one additional axiom needed for this chunk:

$$\mathrm{T1}\colon \forall \mathbf{x}\, \mathbf{x} = \mathtt{St}(\mathbf{x}).$$

Note that the axiom T1 above is actually equivalent to saying that $\forall \mathbf{x}\, \mathtt{Nst}(\mathbf{x}) = 0$.

THE PERMEABILITY RELATION The permeability relation $\rho(S, T)$ is the set of equations of the form

$$Df = g$$

where $f \in E$. The function g which is permeated by this permeability relation will be the first derivative of f in $\mathbb{R}$.

Example 5.6. Suppose that $f(\mathbf{x}) = \mathbf{3x}$ for all $\mathbf{x}$. First, working within Σ_S, the operator D is applied to f such that:

$$\begin{aligned} Df &= \mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) \\ &= \frac{\mathbf{3}(\mathbf{x} + \boldsymbol{\varepsilon}) - \mathbf{3x}}{\boldsymbol{\varepsilon}} \\ &= \frac{\mathbf{3}\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}} = \mathbf{3}. \end{aligned}$$

Note that $\mathtt{St}(\mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})) = \mathtt{St}(\mathbf{3}) = 3 = f'(x)$, and so $f(x) \in E$. Permeating the last equation of Df above to Σ_T gives us:

$$Df = 3$$

and so the derivative of $f(x) = 3x$ is 3.

Example 5.7. Suppose that $f(\mathbf{x}) = \mathbf{x}^2 + \mathbf{2x} + \mathbf{3}$ for all $\mathbf{x}$. First, working within Σ_S, the operator D is applied to f such that:

$$\begin{aligned} Df &= \mathtt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) \\ &= \frac{(\mathbf{x}+\boldsymbol{\varepsilon})^2 + \mathbf{2}(\mathbf{x}+\boldsymbol{\varepsilon}) + \mathbf{3} - \mathbf{x}^2 - \mathbf{2x} - \mathbf{3}}{\boldsymbol{\varepsilon}} \\ &= \frac{\mathbf{2x}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^2 + \mathbf{2}\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}} \\ &= \mathbf{2x} + \boldsymbol{\varepsilon} + \mathbf{2}. \end{aligned}$$

Note that the standard part of $\mathbf{2x} + \boldsymbol{\varepsilon} + \mathbf{2}$ will depend on the domain of $\mathbf{x}$. That is:

$$\mathtt{St}(\mathbf{2x} + \boldsymbol{\varepsilon} + \mathbf{2}) = \begin{cases} 2x + 2, & \text{if } x \in \mathbb{R} \\ 2, & \text{else.} \end{cases}$$

In other words, if (and only if) $\mathtt{Nst}(\mathbf{x}) = 0$, i.e. $x \in \mathbb{R}$, Df can be permeated into Σ_T. Thus, if x is a real number, then we have the derivative of $f(x) = x^2 + 2x + 3 = 2x + 2$.

Example 5.8. Suppose that $f(\mathbf{x}) = \text{sign}(\mathbf{x})$ is defined as:

$$\text{sign}(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathtt{St}(\mathbf{x}) > 0 \\ 0, & \text{if } \mathtt{St}(\mathbf{x}) = 0 \\ -1, & \text{if } \mathtt{St}(\mathbf{x}) < 0 \end{cases}$$

First, working within Σ_S, the operator D is applied to f so that:

$$\begin{aligned} Df &= \texttt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon}) \\ &= \frac{\operatorname{sign}(\mathbf{x}+\boldsymbol{\varepsilon}) - \operatorname{sign}(\mathbf{x})}{\boldsymbol{\varepsilon}} \\ &= 0 \; (\text{because } \forall x \, \texttt{St}(\mathbf{x}) = \texttt{St}(\mathbf{x}+\varepsilon)) \end{aligned}$$

Note that $\texttt{St}(\texttt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})) = \texttt{St}(0) = 0 = f'(x)$, and so $f(x) \in E$. Permeating the last equation of Df above to Σ_T gives us:

$$Df = 0$$

and so the derivative of $f(\mathbf{x}) = \operatorname{sign}(\mathbf{x})$ is 0 for all $\mathbf{x}$. Notice that this is not the case in $\mathbb{R}$, where the derivative of the sign function at $x = 0$ is not defined because of its discontinuity. However, this is not really a bizarre behaviour because if we look very closely at the infinitesimal neighbourhood of $\mathbf{x}$ when $\texttt{St}(\mathbf{x}) = 0$, the function $\operatorname{sign}(\mathbf{x})$ will look like a straight horizontal line and so it makes a perfect sense to have 0 as the slope of the tangent line there. Moreover, this phenomenon also happens in distribution theory where sign function has its derivative everywhere.

5.2 Transcendental Functions in $\mathbb{R}^{\mathbb{Z}_<}$

As we know, there are some special functions defined in real numbers and two of them are the trigonometric and the exponential functions. How then are these functions defined in $\mathbb{R}^{\mathbb{Z}_<}$? Here we propose to define them using power series.

The first two trigonometric functions that we are going to discuss are the sin and cos functions. Using the MacLaurin power series, these two functions are defined as follows:

$$\sin(\mathbf{x}) = \sum_{n=0} \frac{(-1)^n}{(2n+1)!}\mathbf{x}^{2n+1} \tag{5.4}$$

and

$$\cos(\mathbf{x}) = \sum_{n=0} \frac{(-1)^n}{(2n)!}\mathbf{x}^{2n}. \tag{5.5}$$

For the exponential function, we define it as:

$$\exp(\mathbf{x}) = \sum_{n=0} \frac{1}{n!}\mathbf{x}^{n}. \tag{5.6}$$

Note that the MacLaurin polynomial is just a special case of Taylor polynomial with regards to how the function is approximated at $\mathbf{x} = \mathbf{0}$.

Some calculations that might be useful:

1.

$$\begin{aligned}
\texttt{Der}_{\sin \mathbf{x}}(\mathbf{0}, \mathbf{x}) - \texttt{Der}_{\sin \mathbf{x}}(\mathbf{0}, \boldsymbol{\varepsilon}) &= \frac{\sin \mathbf{x}}{\mathbf{x}} - \frac{\sin \boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}} \\
&= \frac{\boldsymbol{\varepsilon} \sin \mathbf{x} - \mathbf{x} \sin \boldsymbol{\varepsilon}}{\mathbf{x}\boldsymbol{\varepsilon}}
\end{aligned}$$

$$= \frac{\left(\boldsymbol{\varepsilon}\mathbf{x} - \frac{1}{3!}\boldsymbol{\varepsilon}\mathbf{x}^3 + \frac{1}{5!}\boldsymbol{\varepsilon}\mathbf{x}^5 - \dots\right) - \boldsymbol{\varepsilon}\mathbf{x} + \frac{1}{3!}\mathbf{x}\boldsymbol{\varepsilon}^3 - \frac{1}{5!}\mathbf{x}\boldsymbol{\varepsilon}^5 + \dots}{\mathbf{x}\boldsymbol{\varepsilon}}$$

$$= \frac{\frac{1}{3!}\mathbf{x}\boldsymbol{\varepsilon}^3 - \frac{1}{3!}\mathbf{x}^3\boldsymbol{\varepsilon} - \frac{1}{5!}\mathbf{x}\boldsymbol{\varepsilon}^5 + \frac{1}{5!}\mathbf{x}^5\boldsymbol{\varepsilon} + \dots}{\mathbf{x}\boldsymbol{\varepsilon}}$$

$$= \frac{1}{3!}\boldsymbol{\varepsilon}^2 - \frac{1}{3!}\mathbf{x}^2 - \frac{1}{5!}\boldsymbol{\varepsilon}^4 + \frac{1}{5!}\mathbf{x}^4 + \dots$$

$$= -\frac{1}{3!}\mathbf{x}^2 + \frac{1}{5!}\mathbf{x}^4 - \frac{1}{5!}\mathbf{x}^6 + \cdots + \frac{1}{3!}\boldsymbol{\varepsilon}^2 - \frac{1}{5!}\boldsymbol{\varepsilon}^4 + \dots$$

$$= \sum_{n=0} \frac{(-1)^{n+1}}{(2n+3)!}\,\mathbf{x}^{2(n+1)} + \sum_{n=0} \frac{(-1)^n}{(2n+3)!}\,\boldsymbol{\varepsilon}^{2(n+1)}.$$

2.

$$\mathtt{Der}_{\sin\mathbf{x}}(\mathbf{x}, \boldsymbol{\varepsilon}) = \frac{\sin(\mathbf{x} + \boldsymbol{\varepsilon}) - \sin\mathbf{x}}{\boldsymbol{\varepsilon}}$$

$$= \frac{\mathbf{x} + \boldsymbol{\varepsilon} - \frac{1}{3!}(\mathbf{x} + \boldsymbol{\varepsilon})^3 + \frac{1}{5!}(\mathbf{x} + \boldsymbol{\varepsilon})^5 - \cdots - \mathbf{x} + \frac{1}{3!}\mathbf{x}^3 - \frac{1}{5!}\mathbf{x}^5 + \dots}{\boldsymbol{\varepsilon}}$$

$$= \frac{\boldsymbol{\varepsilon} + \frac{1}{3!}\left(\mathbf{x}^3 - (\mathbf{x} + \boldsymbol{\varepsilon})^3\right) + \frac{1}{5!}\left(-\mathbf{x}^5 + (\mathbf{x} + \boldsymbol{\varepsilon})^5\right) + \dots}{\boldsymbol{\varepsilon}}$$

$$= \frac{\boldsymbol{\varepsilon} + \frac{1}{3!}\left(-3\mathbf{x}^2\boldsymbol{\varepsilon} - 3\mathbf{x}\boldsymbol{\varepsilon}^2 - \boldsymbol{\varepsilon}^3\right) + \frac{1}{5!}\left(5\mathbf{x}^4\boldsymbol{\varepsilon} + 10\mathbf{x}^3\boldsymbol{\varepsilon}^2 + 10\mathbf{x}^2\boldsymbol{\varepsilon}^3 + 5\mathbf{x}\boldsymbol{\varepsilon}^4 + \boldsymbol{\varepsilon}^5\right) + \dots}{\boldsymbol{\varepsilon}}$$

$$= 1 - \frac{1}{2!}\mathbf{x}^2 - \frac{1}{2!}\mathbf{x}\boldsymbol{\varepsilon} - \frac{1}{3!}\boldsymbol{\varepsilon}^2 + \frac{1}{4!}\mathbf{x}^4 + \frac{1}{4!}2\mathbf{x}^3\boldsymbol{\varepsilon} + \frac{1}{5!}2\mathbf{x}^2\boldsymbol{\varepsilon}^2 + \frac{1}{4!}\mathbf{x}\boldsymbol{\varepsilon}^3 + \frac{1}{5!}\boldsymbol{\varepsilon}^4 + \dots$$

$$= 1 - \frac{1}{2!}\mathbf{x}^2 + \frac{1}{4!}\mathbf{x}^4 + \cdots - \frac{1}{2!}\mathbf{x}\boldsymbol{\varepsilon} - \frac{1}{3!}\boldsymbol{\varepsilon}^2 + \frac{1}{4!}2\mathbf{x}^3\boldsymbol{\varepsilon} + \frac{1}{5!}2\mathbf{x}^2\boldsymbol{\varepsilon}^2 + \frac{1}{4!}\mathbf{x}\boldsymbol{\varepsilon}^3 + \frac{1}{5!}\boldsymbol{\varepsilon}^4 + \dots$$

$$= \cos\mathbf{x} - \frac{1}{2!}\mathbf{x}\boldsymbol{\varepsilon} - \frac{1}{3!}\boldsymbol{\varepsilon}^2 + \frac{1}{4!}2\mathbf{x}^3\boldsymbol{\varepsilon} + \frac{1}{5!}2\mathbf{x}^2\boldsymbol{\varepsilon}^2 + \frac{1}{4!}\mathbf{x}\boldsymbol{\varepsilon}^3 + \frac{1}{5!}\boldsymbol{\varepsilon}^4 + \dots$$

$= \cos \mathbf{x} + R(\mathbf{x}, \boldsymbol{\varepsilon})\boldsymbol{\varepsilon}$, where $R(\mathbf{x}, \boldsymbol{\varepsilon})$ is the reminder function.

Note that $\texttt{St}(\texttt{Der}_f(\mathbf{x}, \boldsymbol{\varepsilon})) = \cos \mathbf{x} = f'(x)$, and so $f(\mathbf{x}) \in E$. Permeating the last equation of Df above to Σ_T gives us:

$$Df = \cos x$$

and so the derivative of $f(\mathbf{x}) = \sin \mathbf{x}$ is $\cos \mathbf{x}$, es expected.

Example 5.9. Suppose that we have $\mathbf{x} = x + a\boldsymbol{\varepsilon} = \langle \widehat{x}, a, 0, 0, \ldots \rangle$ where $x, a \in \mathbb{R}$. We want to know what $\sin(\mathbf{x})$ is. Based on Equation (5.4),

$$\sin(\mathbf{x}) = \sin(x + \boldsymbol{\varepsilon}) = (x + \boldsymbol{\varepsilon}) - \tfrac{1}{3!}(x + \boldsymbol{\varepsilon})^3 + \tfrac{1}{5!}(x + \boldsymbol{\varepsilon})^5 - \tfrac{1}{7!}(x + \boldsymbol{\varepsilon})^7 + \ldots$$

Our task now is to find all the members of $\texttt{Nst}_{\varepsilon}(\sin(\mathbf{x}))$ and also $\texttt{St}(\sin(\mathbf{x}))$, which are shown in Table 5.2.1. Note that from the way we define sin function, $x_i = 0\ \forall x_i \in \texttt{Nst}_{\omega}(\sin(\mathbf{x}))$. Thus from Table 5.2.1, we get:

$$\begin{aligned}\sin(\mathbf{x}) &= \sin(x + a\boldsymbol{\varepsilon}) \\ &= \langle \widehat{\sin(x)}, a\cos(x), -\tfrac{a^2}{2!}\sin(x), -\tfrac{a^3}{3!}\cos(x), \tfrac{a^4}{4!}\sin(x), \tfrac{a^5}{5!}\cos(x), \ldots \rangle,\end{aligned}$$

and we also get

$$\sin(\boldsymbol{\varepsilon}) = \langle \widehat{0}, 1, 0, -\tfrac{1}{3!}, 0, \tfrac{1}{5!}, \ldots \rangle = \boldsymbol{\varepsilon} - \tfrac{1}{3!}\boldsymbol{\varepsilon}^3 + \tfrac{1}{5!}\boldsymbol{\varepsilon}^5 - \ldots$$

for an infinitesimal angle $\boldsymbol{\varepsilon}$.

	Expanded Form	Simplified Form
real-part	$= \mathbf{x} - \frac{1}{3!}\mathbf{x}^3 + \frac{1}{5!}\mathbf{x}^5 - \frac{1}{7!}\mathbf{x}^7 + \dots$ $= \sum_{n=0} \frac{-1^n}{(2n+1)!}\,\mathbf{x}^{2n+1}$	$= \sin(\mathbf{x})$
$\boldsymbol{\varepsilon}$-part	$= a\boldsymbol{\varepsilon} - \frac{1}{2!}\mathbf{x}^2 a\boldsymbol{\varepsilon} + \frac{1}{4!}a\boldsymbol{\varepsilon}\mathbf{x}^4 - \frac{1}{6!}a\boldsymbol{\varepsilon}\mathbf{x}^6 + \dots$ $= \boldsymbol{\varepsilon}(a - \frac{1}{2!}a\mathbf{x}^2 + \frac{1}{4!}a\mathbf{x}^4 - \frac{1}{6!}a\mathbf{x}^6 + \dots)$ $= \boldsymbol{\varepsilon}\sum_{n=0} \frac{-1^n}{(2n)!}\,a\mathbf{x}^{2n}$	$= \boldsymbol{\varepsilon}(a\cos(\mathbf{x}))$
$\boldsymbol{\varepsilon}^2$-part	$= -\frac{3}{3!}a^2\boldsymbol{\varepsilon}^2\mathbf{x} + \frac{10}{5!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^3 - \frac{21}{7!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^5 + \dots$ $= -\frac{1}{2!}a^2\boldsymbol{\varepsilon}^2\mathbf{x} + \frac{2}{4!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^3 - \frac{3}{6!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^5 + \dots$ $= \boldsymbol{\varepsilon}^2(-\frac{1}{2!}\mathbf{x} + \frac{2}{4!}a^2\mathbf{x}^3 - \frac{3}{6!}a^2\mathbf{x}^5 + \dots)$ $= \boldsymbol{\varepsilon}^2\sum_{n=0} \frac{-1^{n+1}(n+1)}{(2n+2)!}\,a^2\mathbf{x}^{2n+1}$	$= \boldsymbol{\varepsilon}^2(-\frac{a^2}{2}\sin(\mathbf{x}))$
$\boldsymbol{\varepsilon}^3$-part	$= -\frac{1}{3!}a^3\boldsymbol{\varepsilon}^3 + \frac{10}{5!}a^3\boldsymbol{\varepsilon}^3\mathbf{x}^2 - \frac{35}{7!}a^3\boldsymbol{\varepsilon}^3\mathbf{x}^4 + \frac{84}{9!}a^3\boldsymbol{\varepsilon}^3\mathbf{x}^6 - \dots$ $= \boldsymbol{\varepsilon}^3(-\frac{1}{3!}a^3\mathbf{x}^0 + \frac{10}{5!}a^3\mathbf{x}^2 - \frac{35}{7!}a^3\mathbf{x}^4 + \frac{84}{9!}a^3\mathbf{x}^6 - \dots)$ $= \boldsymbol{\varepsilon}^3\sum_{n=0} \frac{-1^{n+1}}{6(2n)!}\,a^3\mathbf{x}^{2n}$	$= \boldsymbol{\varepsilon}^3(-\frac{a^3}{6}\cos(\mathbf{x}))$
$\boldsymbol{\varepsilon}^4$-part	$= \frac{5}{5!}a^4\boldsymbol{\varepsilon}^4\mathbf{x} - \frac{35}{7!}a^4\boldsymbol{\varepsilon}^4\mathbf{x}^3 + \frac{126}{9!}a^4\boldsymbol{\varepsilon}^4\mathbf{x}^5 - \frac{330}{11!}a^4\boldsymbol{\varepsilon}^4\mathbf{x}^7 + \dots$ $= \boldsymbol{\varepsilon}^4(\frac{1}{4!}\mathbf{x} - \frac{5}{6!}a^4\mathbf{x}^3 + \frac{14}{8!}a^4\mathbf{x}^5 - \frac{30}{10!}a^4\mathbf{x}^7 + \dots)$ $= \boldsymbol{\varepsilon}^4\sum_{n=0} \frac{-1^n}{24(2n+1)!}\,a^4\mathbf{x}^{2n+1}$	$= \boldsymbol{\varepsilon}^4(\frac{a^4}{24}\sin(\mathbf{x}))$

Table 5.2.1: $\texttt{St}(\sin(\mathbf{x}))$ and the first four members of $\texttt{Nst}_{\varepsilon}(\sin(\mathbf{x}))$

real-part	$= 1 - \frac{1}{2!}\mathbf{x}^2 + \frac{1}{4!}\mathbf{x}^4 - \frac{1}{6!}\mathbf{x}^6 + \ldots$ $= \sum_{n=0} \frac{-1^n}{(2n)!}\mathbf{x}^{2n}$	$= \cos(\mathbf{x})$
$\boldsymbol{\varepsilon}$-part	$= -\frac{2}{2!}a\boldsymbol{\varepsilon}\mathbf{x} + \frac{4}{4!}a\boldsymbol{\varepsilon}\mathbf{x}^3 - \frac{6}{6!}a\boldsymbol{\varepsilon}\mathbf{x}^5 + \ldots$ $= \boldsymbol{\varepsilon}(-a\mathbf{x} + \frac{1}{3!}a\mathbf{x}^3 - \frac{1}{5!}a\mathbf{x}^5 + \ldots)$ $= \boldsymbol{\varepsilon}\sum_{n=0} \frac{-1^{n+1}}{(2n+1)!} a\mathbf{x}^{2n+1}$	$= \boldsymbol{\varepsilon}(-a\sin(\mathbf{x}))$
$\boldsymbol{\varepsilon}^2$-part	$= -\frac{1}{2!}a^2\boldsymbol{\varepsilon}^2 + \frac{6}{4!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^2 - \frac{15}{6!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^4 + \frac{28}{8!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^6 - \ldots$ $= \boldsymbol{\varepsilon}^2\sum_{n=0} \frac{-1^{n+1}(n+1)(2n+1)}{(2n+2)!} a^2\mathbf{x}^{2n}$ $= \boldsymbol{\varepsilon}^2\sum_{n=0} \frac{1}{2}\frac{-1^{n+1}}{(2n)!}\mathbf{x}^{2n}$	$= \boldsymbol{\varepsilon}^2(-\frac{a^2}{2}\cos(\mathbf{x}))$

Table 5.2.2: $\mathtt{St}(\cos(\mathbf{x}))$ and the first two members of $\mathtt{Nst}_{\varepsilon}(\cos(\mathbf{x}))$

Example 5.10. Again, suppose that we have $\mathbf{x} = x + a\varepsilon$ where $x, a \in \mathbb{R}$. Here we try to find what $\cos\mathbf{x}$ is. With a similar method to that used in Example 5.9, we have a calculation like that shown in Table 5.2.2.

Thus from Table 5.2.2, we get:

$$\cos(\mathbf{x}) = \cos(x + a$$

$$mathbfepsilon) = \langle\widehat{\cos(x)}, -a\sin(x), -\frac{a^2}{2!}\cos(x), \frac{a^3}{3!}\sin(x), \frac{a^4}{4!}\cos(x), \ldots\rangle,$$

and we also get

$$\cos(\boldsymbol{\varepsilon}) = \langle\widehat{1}, 0, -\frac{1}{2}, 0, \ldots\rangle = 1 - \frac{1}{2!}\boldsymbol{\varepsilon}^2 + \ldots$$

for an infinitesimal angle $\boldsymbol{\varepsilon}$.

Example 5.11. With the same $\mathbf{x}$ as in Examples 5.9 and 5.10, we try to know what $\exp(\mathbf{x})$ is. Based on Equation (5.6),

$$\exp(\mathbf{x}) = \exp(x + a\boldsymbol{\varepsilon}) = 1 + (x + a\boldsymbol{\varepsilon}) - \frac{1}{2!}(x + a\boldsymbol{\varepsilon})^2 + \frac{1}{3!}(x + a\boldsymbol{\varepsilon})^3 + \ldots$$

	Expanded Form	Simplified Form
real-part	$= 1 + \mathbf{x} + \frac{1}{2!}\mathbf{x}^2 + \frac{1}{3!}\mathbf{x}^3 + \ldots$ $= \sum_{n=0} \frac{1}{n!} \mathbf{x}^n$	$= \exp(\mathbf{x})$
$\boldsymbol{\varepsilon}$-part	$= a\boldsymbol{\varepsilon} + \frac{2}{2!}a\boldsymbol{\varepsilon}\mathbf{x} + \frac{3}{3!}a\boldsymbol{\varepsilon}\mathbf{x}^2 + \frac{4}{4!}a\boldsymbol{\varepsilon}\mathbf{x}^3\boldsymbol{\varepsilon} + \ldots$ $= \boldsymbol{\varepsilon}(a + a\mathbf{x} + \frac{1}{2!}a\mathbf{x}^2 + \frac{1}{3!}a\mathbf{x}^3 - \frac{1}{4!}a\mathbf{x}^4 + \ldots)$ $= \boldsymbol{\varepsilon} \sum_{n=0} \frac{1}{n!} a\mathbf{x}^n$	$= \boldsymbol{\varepsilon}(a \exp(\mathbf{x}))$
$\boldsymbol{\varepsilon}^2$-part	$= \frac{1}{2!}a^2\boldsymbol{\varepsilon}^2\mathbf{x} + \frac{3}{3!}a^2\boldsymbol{\varepsilon}^2\mathbf{x} + \frac{6}{4!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^2 + \frac{10}{5!}a^2\boldsymbol{\varepsilon}^2\mathbf{x}^3 + \ldots$ $= \boldsymbol{\varepsilon}^2(\frac{1}{2!}a^2 + \frac{3}{3!}a^2\mathbf{x} + \frac{6}{4!}a^2\mathbf{x}^2 + \frac{10}{5!}a^2\mathbf{x}^3 + \ldots)$ $= \boldsymbol{\varepsilon}^2 \sum_{n=0} \frac{(n+1)(n+2)}{2(n+2)!} a^2\mathbf{x}^n$ $= \boldsymbol{\varepsilon}^2 \sum_{n=0} \frac{1}{2(n!)} a^2\mathbf{x}^n$	$= \boldsymbol{\varepsilon}^2(\frac{a^2}{2} \exp(\mathbf{x}))$
$\boldsymbol{\varepsilon}^3$-part	$= \frac{1}{3!}a^3\boldsymbol{\varepsilon}^3 + \frac{4}{4!}a^3\boldsymbol{\varepsilon}^3\mathbf{x} + \frac{10}{5!}a^3\boldsymbol{\varepsilon}^3\mathbf{x}^2 + \frac{20}{6!}a^3\boldsymbol{\varepsilon}^3\mathbf{x}^3 + \frac{35}{7!}a^3\boldsymbol{\varepsilon}^3\mathbf{x}^4 + \ldots$ $= \boldsymbol{\varepsilon}^3(\frac{1}{3!}a^3 + \frac{4}{4!}a^3\mathbf{x} - \frac{10}{5!}a^3\mathbf{x}^2 + \frac{20}{6!}a^3\mathbf{x}^3 + \frac{35}{7!}a^3\mathbf{x}^4 + \ldots)$ $= \boldsymbol{\varepsilon}^3 \sum_{n=0} \frac{(n+1)(n+2)(n+3)}{6(n+3)!} a^3\mathbf{x}^n$ $= \boldsymbol{\varepsilon}^3 \sum_{n=0} \frac{1}{6(n!)} a^3\mathbf{x}^n$	$= \boldsymbol{\varepsilon}^3(\frac{a^3}{6} \exp(\mathbf{x}))$

Table 5.2.3: $\mathtt{St}(\exp(\mathbf{x}))$ and The First Three Members of $\mathtt{Nst}_\varepsilon(\exp(\mathbf{x}))$

Our task now is to find all the members of $\mathtt{Nst}_\varepsilon(\exp(\mathbf{x}))$ and also $\mathtt{St}(\exp(\mathbf{x}))$, which are shown in Table 5.2.3. Note that from the way we define the function exp, $\forall x_i \in \mathtt{Nst}_\omega(\exp(\mathbf{x}))\ x_i = 0$. Thus from Table 5.2.3, we get:

$$\exp(\mathbf{x}) = \exp(x + a\boldsymbol{\varepsilon}) = \langle \widehat{\exp(x)}, a\exp(x), \tfrac{a^2}{2!}\exp(x), \tfrac{a^3}{3!}\exp(x), \tfrac{a^4}{4!}\exp(x), \ldots \rangle,$$

and we also get

$$\exp(\boldsymbol{\varepsilon}) = \langle \widehat{1}, 1, \tfrac{1}{2!}, \tfrac{1}{3!}, \tfrac{1}{4!}, \ldots \rangle = 1 + \boldsymbol{\varepsilon} + \tfrac{1}{2!}\boldsymbol{\varepsilon}^2 + \tfrac{1}{3!}\boldsymbol{\varepsilon}^3 - \ldots$$

for an infinitesimal angle $\boldsymbol{\varepsilon}$.

From the preceding discussion, we have the following proposition.

Proposition 5.12. *For the* sin, cos, *and* exp *functions:*

1. $\mathtt{Der}_{\sin\mathbf{x}}(\mathbf{x},\boldsymbol{\varepsilon}) = \frac{\sin(\mathbf{x}+\boldsymbol{\varepsilon})-\sin(\mathbf{x})}{\boldsymbol{\varepsilon}} = \langle\widehat{\cos(x)}, -\frac{1}{2!}\sin(x), -\frac{1}{3!}\cos(x),\ldots\rangle$, *and so we have:*

$$\mathtt{St}(\mathtt{Der}_{\sin(\mathbf{x})}(\mathbf{x},\boldsymbol{\varepsilon})) = \cos(\mathbf{x})$$

2. $\mathtt{Der}_{\cos\mathbf{x}}(\mathbf{x},\boldsymbol{\varepsilon}) = \frac{\cos(\mathbf{x}+\boldsymbol{\varepsilon})-\cos(\mathbf{x})}{\boldsymbol{\varepsilon}} = \langle\widehat{-\sin(x)}, -\frac{1}{2!}\cos(x), \frac{1}{3!}\sin(x),\ldots\rangle$, *and so*

$$\mathtt{St}(\mathtt{Der}_{\cos(\mathbf{x})}(\mathbf{x},\boldsymbol{\varepsilon})) = -\sin(\mathbf{x})$$

3. $\mathtt{Der}_{\exp\mathbf{x}}(\mathbf{x},\boldsymbol{\varepsilon}) = \frac{\exp(\mathbf{x}+\boldsymbol{\varepsilon})-\exp(\mathbf{x})}{\boldsymbol{\varepsilon}} = \langle\widehat{\exp(x)}, \frac{1}{2!}\exp(x), \frac{1}{3!}\exp(x),\ldots\rangle$, *and so*

$$\mathtt{St}(\mathtt{Der}_{\exp(\mathbf{x})}(\mathbf{x},\boldsymbol{\varepsilon})) = \exp(\mathbf{x})$$

5.3 Anti-derivative

Suppose that a function $f:\mathbb{R}^{\mathbb{Z}_<}\to\mathbb{R}^{\mathbb{Z}_<}$ is defined. In relation to its anti-derivative, what is $\int_0^1 f\,\mathrm{d}x$? By using the same formal concept as that of the usual Riemann integral, we have:

$$\int_0^1 f\,\mathrm{d}x = \sum_{n=1}^{\omega}\varepsilon f(\frac{n}{\omega}) = \sum_{n=1}^{\omega}\varepsilon f(n\varepsilon).$$

One important assumption that we have to make in order to make these calculations below work is:

$$\text{If}\sum_{n=0}^{k}a(n) = A(k),\text{ then}\sum_{n=0}^{\omega}a(n) = A(\omega), \tag{5.7}$$

where $n, k \in \mathbb{N}$ and ω is an infinite number in $\mathbb{R}^{\mathbb{Z}_<}$.

Example 5.13. For $f(x) = x$, we have

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{n=1}^{\omega} \varepsilon(n\varepsilon) = \varepsilon^2 \sum_{n=1}^{\omega} n = \varepsilon^2 \tfrac{1}{2}(\omega)(\omega+1) = \tfrac{1}{2} + \tfrac{1}{2}\varepsilon.$$

If we want to have its standard part, then we get

$$\mathtt{St}(\int_0^1 f(x)\,\mathrm{d}x) = \mathtt{St}(\tfrac{1}{2} + \tfrac{1}{2}\varepsilon) = \tfrac{1}{2}.$$

Theorem 5.14. *Suppose that f is the derivative of F. If f has its anti-derivative on* $[0, 1]$ *then*

$$\mathtt{St}\left(\int_0^1 f(x)\,\mathrm{d}x\right) = F(1) - F(0).$$

Proof:

$$\begin{aligned}
\mathtt{St}\left(\int_0^1 f(x)\,\mathrm{d}x\right) &= \mathtt{St}\left(\int_0^1 \frac{F(x+\varepsilon) - F(x)}{\varepsilon}\,\mathrm{d}x\right) \\
&= \mathtt{St}\left(\sum_{n=1}^{\omega} F(n\varepsilon + \varepsilon) - F(n\varepsilon)\right) \\
&= \mathtt{St}\left(\sum_{n=1}^{\omega} F((n+1)\varepsilon) - F(n\varepsilon)\right) \\
&= \mathtt{St}(F(2\varepsilon) - F(\varepsilon) + F(3\varepsilon) - F(2\varepsilon) + F(4\varepsilon) - F(3\varepsilon) + \\
&\quad F(5\varepsilon) - F(4\varepsilon) + \cdots + F(\omega + \varepsilon) - F((\omega - 1)\varepsilon) + \\
&\quad F((\omega + 1)\varepsilon) - F(\omega\varepsilon)) \\
&= \mathtt{St}(F(\omega\varepsilon + \varepsilon) - F(\varepsilon)) \\
&= \mathtt{St}(F(1 + \varepsilon) - F(\varepsilon)) \\
&= F(1) - F(0)
\end{aligned}$$

$\blacksquare$

5.4 Continuity

In this section, we try to pinpoint what the good definition for continuous functions is. We also decide whether we can permeate it between $\mathbb{R}$ and $\mathbb{R}^{\mathbb{Z}_<}$. Note that if the domain and codomain of a function is not explicitly stated, they will be determined from the specified model.

Definition 5.15 ($\mathrm{ED}_{\mathrm{CLASS}}$)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $a \in \mathbb{R}$ if, given $n \in \mathbb{N}$, there exists a $m \in \mathbb{N}$ such that

$$|f(x) - f(a)| < \tfrac{1}{n} \text{ whenever } |x - a| < \tfrac{1}{m}.$$

The function f is called continuous on an interval I iff f is continuous at every point in I.

Definition 5.16 (ED)**.** A function f is continuous at a point $\mathbf{c} \in \mathbb{R}^{\mathbb{Z}_<}$ if, given $\mathfrak{e}_1 \in \mathbb{R}^{\mathbb{Z}_<} > \mathbf{0}$, there exists a $\mathfrak{e}_2 \in \mathbb{R}^{\mathbb{Z}_<} > \mathbf{0}$ such that

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{c})| < \mathfrak{e}_1 \text{ whenever } |\mathbf{x} - \mathbf{c}| < \mathfrak{e}_2.$$

That function $\mathbf{f}$ is called continuous function over an interval I iff $\mathbf{f}$ is continuous at every point in I.

Proposition 5.17. *There exists a function f in $\mathbb{R}^{\mathbb{Z}_<}$ which is continuous under Definition 5.16, but discontinuous under Definition 5.15, i.e.*

$$\mathbb{R}^{\mathbb{Z}_<} \models \exists f \text{ s.t. } (\mathrm{ED}_{\mathrm{CLASS}}(f) \land \neg\mathrm{ED}(f)).$$

PROOF: Suppose that $\Delta = \{x \mid \forall n \in \mathbb{N}, |x| < \frac{1}{n}\}$ – in other words, Δ is a set of all infinitesimals – and consider the indicator function around Δ, that is

$$1_\Delta(x) = \begin{cases} 1, & x \in \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathrm{ED}_{\mathrm{CLASS}}(1_\Delta)$ but $\neg\mathrm{ED}(1_\Delta)$. ∎

Remark 5.18. Note that:

1. The set Δ in $\mathbb{R}$ only have 0 as its member. That is $\mathbb{R} \models \Delta = \{0\}$.
2. In $\mathbb{R}$, both Definitions 5.15 and 5.16 are equivalent, that is for any function f, $\mathbb{R} \models \mathrm{ED}_{\mathrm{CLASS}}(f) \leftrightarrow \mathrm{ED}(f)$.

Property 5.19 (EVP)**.** If I is an interval and $f : I \to J$, we say that f has the extreme value property iff f has its maximum value on I. That is,

$$\forall a \leq b \in I, \exists x \in [a, b] \text{ s.t. } \forall y \in [a, b](f(y) \leq f(x)).$$

Property 5.20 (IVP)**.** If I is an interval, and $f : I = [a, b] \to J$, we say that f has the intermediate value property iff

$$\forall c' \in (f(a), f(b)), \exists c \in (a, b) \text{ s.t. } f(c) = c'.$$

Theorem 5.21. $\mathbb{R} \models \mathrm{ED} \to \mathrm{EVP}$

PROOF: The proof of this theorem can be found in any standard book for Analysis course ([3] for example). ∎

Theorem 5.22. *There is a function f such that*

$$\mathbb{R}^{\mathbb{Z}_<} \models \mathrm{ED}(f) \wedge \neg \mathrm{EVP}(f).$$

Proof: Take the function f on $[1, 2]$ as defined below:

$$f(x) = \begin{cases} \frac{1}{n} & x \sim \frac{m}{n} \text{ (reduced fraction)} \\ 0 & \text{otherwise.} \end{cases}$$

■

Remark 5.23. 'This research now reaches an especially engrossing object. The function $f(x)$ in Theorem 5.22 can be used to construct a fractal-like object. Fractals are classically defined as geometric objects that exhibit some form of self-similarity. Figure 5.4.1 shows what the function $f(x)$ in Theorem 5.22 looks like, and also what occurs when we zoom in on a particular point. In this sense, the function from Theorem 5.22 is an *infinitesimal fractal*. Formally speaking, suppose that we have a function $f : \mathbb{R}^{\mathbb{Z}_<} \to \mathbb{R}$ and let us define another function

$$F : \mathbb{R}^{\mathbb{Z}_<} \to \mathbb{R}^{\mathbb{Z}_<}$$

by

$$\begin{aligned} F(x) &= f(x) + \varepsilon f(x) + \varepsilon^2 f(x) + \dots \\ &= \langle \widehat{f(x)}, f(x), f(x), \dots \rangle. \end{aligned}$$

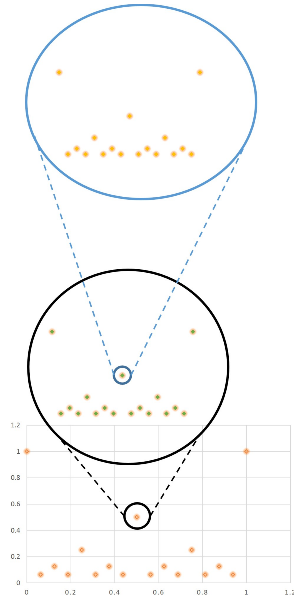


Figure 5.4.1: Illustration of the infinitesimal fractal from the function defined in Theorem 5.22

Then, that function $F(x)$ will define an *infinite fractal* (if $\mathtt{ni}_\omega(\frac{1}{\varepsilon}\widehat{\times}F(x)) = F(x)$) or *infinitesimal fractals* (if $\mathtt{ni}_\omega(\varepsilon\widehat{\times}F(x)) = F(x)$), where $\mathtt{ni}_\omega(\mathbf{x})$ denotes the non-infinity part of $\mathbf{x}$.

Theorem 5.24. $\mathbb{R} \models \mathrm{ED} \to \mathrm{IVP}$.

PROOF: The proof of this theorem can be found in any standard course book of Analysis ([3] for example). ■

Theorem 5.25. *There exists a function f such that*

$$\mathbb{R}^{\mathbb{Z}_<} \models \mathrm{ED}(f) \wedge \neg\mathrm{IVP}(f).$$

PROOF: Take the function 1_Δ as defined in Proposition 5.17. This function satisfies Definition 5.16. Now take two numbers -7 and $\boldsymbol{\varepsilon}$. It is clear that

$-7 < \boldsymbol{\varepsilon}$, $1_\Delta(-7) = 0$, $1_\Delta(\boldsymbol{\varepsilon}) = 1$ and we have $\frac{1}{2}$, for example, between **0** and 1 but there does not exist any d between -7 and $\boldsymbol{\varepsilon}$ such that $1_\Delta(d) = \frac{1}{2}$. Thus, the function 1_Δ does not satisfy Property 5.20. ■

Theorem 5.26. *There exists a function f such that*

$$\mathbb{R} \models \mathrm{EVP}(f) \wedge \neg \mathrm{ED}_{\mathrm{CLASS}}(f).$$

Proof: Take the function $1_\mathbb{Q}$ as defined below:

$$1_\mathbb{Q} = \begin{cases} 1 & \mathbf{x} \in \mathbb{Q} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

■

Theorem 5.27.

Theorem 5.28. *There exists a function f such that*

$$\mathbb{R} \models \mathrm{EVP}(f) \wedge \neg \mathrm{IVP}(f).$$

Proof: Take the function f on $[0, 1]$ defined as:

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 5 & \frac{1}{2} < x \leq 1. \end{cases}$$

That function f satisfies $\mathrm{EVP}_\mathbb{R}$ (as it attains its maximum value at 5), but does not satisfy $\mathrm{IVP}_\mathbb{R}$ because take 3, for example, between $f(0)$ and $f(1)$, there is no d

between 0 and 1 such that $f(d) = 3$. ■

Theorem 5.29. *There exists a function f such that*

$$\mathbb{R}^{\mathbb{Z}_<} \models \mathrm{EVP}(f) \wedge \neg\mathrm{IVP}(f).$$

Proof: Take the same function as in Theorem 5.28. ■

Theorem 5.30. *There exists a function f such that*

$$\mathbb{R} \models \mathrm{IVP}(f) \wedge \neg\mathrm{EVP}(f).$$

Proof: Take the function $f(x)$ defined as:

$$f(x) = \begin{cases} (1-x) \cdot \sin(\frac{1}{x}) & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

This function f satisfies $\mathrm{IVP}_{\mathbb{R}}$ but not $\mathrm{EVP}_{\mathbb{R}}$. ■

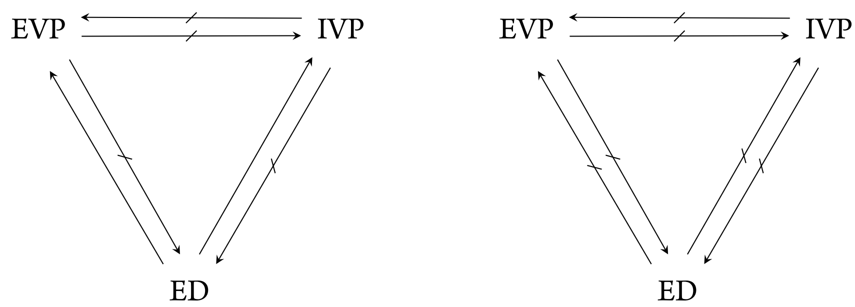
Theorem 5.31. *There exists a function f such that*

$$\mathbb{R}^{\mathbb{Z}_<} \models \mathrm{IVP}(f) \wedge \neg\mathrm{EVP}(f).$$

Proof: Take the function f as defined as:

$$f(\mathbf{x}) = \begin{cases} \frac{\mathbf{1}}{\mathbf{x}} & \mathbf{0} < \mathbf{x} \leq \mathbf{1} \\ \mathbf{0} & \mathbf{x} = \mathbf{0}. \end{cases}$$

This function f satisfies $\mathrm{IVP}_{\mathbb{R}^{\mathbb{Z}_<}}$ but not $\mathrm{EVP}_{\mathbb{R}^{\mathbb{Z}_<}}$. ■



(a) In $\mathbb{R}$ **(b)** In $\mathbb{R}^{\mathbb{Z}_<}$

Figure 5.4.2: Relationship among three definitions of continuity

Theorem 5.32. *There exists a function f such that*

$$\mathbb{R} \models \mathrm{IVP}(f) \wedge \neg \mathrm{ED}(f).$$

PROOF: Take the Darboux function:

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

The Darboux function above satisfies $\mathrm{IVP}_{\mathbb{R}}$ but does not satisfy $\mathrm{ED}_{\mathbb{R}}$. ■

So now, the obvious question worth asking is how do we define continuity in our set $\mathbb{R}^{\mathbb{Z}_<}$? As we can see, there are three possible ways to define it, namely: with the ε-δ definition (ED), with extreme value property (EVP), or with the intermediate value property (IVP). We will now discuss them one by one.

Firstly, through IVP. What does IVP actually say. Basically, it says that for every value within the range of the given function, we can find a point in the domain corresponding to that value. Will this work in our set? Let us consider the

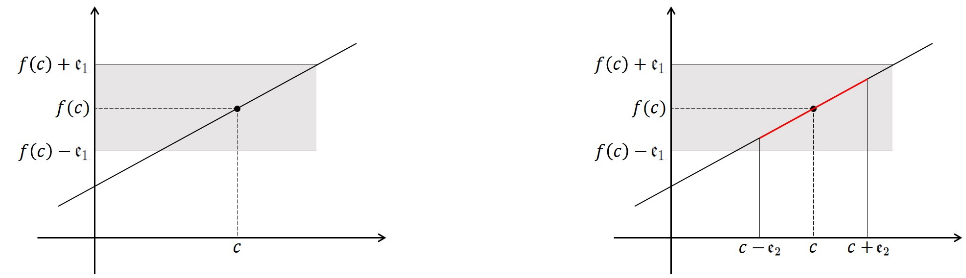
$\mathbb{R}^{\mathbb{Z}_<}$-valued function $f(x) = x^2$ on $[a, b]$ for any $a, b \in \mathbb{R}^{\mathbb{Z}_<}$ and let us assume that IVP holds. It follows that for every c' between $f(a) = a^2$ and $f(b) = b^2$, $\exists c \in (a, b)$ s.t. $f(c) = c^2 = c'$. The only c which satisfies that last equation is $c = \sqrt{c'}$, which cannot be defined in our set $\mathbb{R}^{\mathbb{Z}_<}$. Thus, IVP, even though it is somehow intuitively "obvious", it does not really work in $\mathbb{R}^{\mathbb{Z}_<}$. This phenomenon is actually not uncommon if we want to have a world with infinitesimals (or infinities) in it. See [5, p. 107] for example.

However, note that in $\mathbb{R}$, the function x^2 still satisfies IVP. Now, is there a function in $\mathbb{R}^{\mathbb{Z}_<}$ that satisfies IVP? Consider the identity function $f(x) = x$. This function clearly satisfies IVP in both domains, and so we have the following theorem.

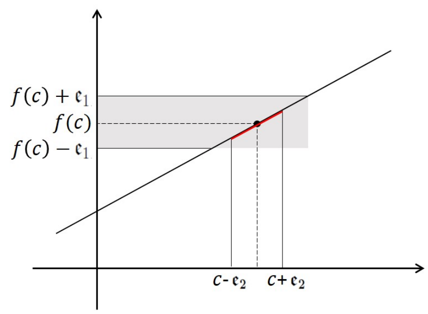
Theorem 5.33. *There exists a function f such that* $(\mathbb{R} \models \mathrm{IVP}(f) \wedge \mathbb{R}^{\mathbb{Z}_<} \models \neg\mathrm{IVP}(f))$*, and there exists a function g s.t.* $\mathbb{R}, \mathbb{R}^{\mathbb{Z}_<} \models \mathrm{IVP}(g)$.

Hence, from the argument above we argue that defining continuity in our set with IVP is not really useful.

Secondly, in regards to EVP. This is clearly not a good way to define continuity in our set because even in the set of real numbers, there are some continuous functions which do not satisfy EVP themselves. So the last available option now is the third one, which is the ε-δ (ED) definition. We argue that this definition is the best way to define continuity in $\mathbb{R}^{\mathbb{Z}_<}$. Moreover, in this way, it preserves much of the spirit of classical analysis on $\mathbb{R}$ while retaining the intuition of infinitesimals.



(a) An $\mathfrak{e}_1$ bound & its $\mathfrak{e}_2$ neighbourhood fulfilling Definition 5.16



(b) A smaller bound & its neighbourhood

Figure 5.4.3: Illustration of $\mathfrak{e}_1$ and $\mathfrak{e}_2$ intervals

It is important to note that in the ED definition of continuity (Definition 5.16), there are two variables which are in play, i.e. $\mathfrak{e}_1$ and $\mathfrak{e}_2$. When we apply this definition on our set, these two variables hold important (or rather, very interesting) roles where we will have different levels of continuity from the same function. What we mean is that these two variables can greatly vary depending on how far (deep) we want to push (observe) them, e.g. $\mathfrak{e}_2$ can be a real number ($\mathfrak{e}_2 \in \Delta^{\circ}$), or it can be in Δ^4, Δ^8 and so on. Remember that these two numbers, $\mathfrak{e}_1$ and $\mathfrak{e}_2$, will determine how subtle we want our intervals to be (see Figure 5.4.3 for illustration).

Thus this definition of continuity works as follows. Suppose that we have a function f and we want to decide whether it is continuous or not. With this concept of two variables, we will have what we call as k, n-continuity where $k, n \in \mathbb{N} \cup \{0\}$.

Definition 5.34 (k, n-Continuity)**.** A function f is k, n-continuous at a point $\mathbf{c}$ iff $\forall \mathfrak{e}_{1_\mathbf{k}} > 0, \exists \mathfrak{e}_{2_n} > 0$ such that

$$\text{if } |\mathbf{x} \widehat{-} \mathbf{c}| < \mathfrak{e}_{2_\mathbf{n}}, \text{ then } |\mathbf{f}(\mathbf{x}) \widehat{-} \mathbf{f}(\mathbf{c})| < \mathfrak{e}_{1_\mathbf{k}}$$

where $\mathfrak{e}_{1_p}, \mathfrak{e}_{2_p} \in \Delta^p$.

Definition 5.35. A function f is said to be k, n-continuous iff it is (k, n)-continuous at every point in the given domain.

Remark 5.36. From the definition of the set Δ^m, note that

$$\text{for any } r \in \mathbb{R}^{\mathbb{Z}_<}, d \in \Delta^p, \text{ and } e \in \Delta^{p+1}, (r - e, r + e) \subseteq (r - d, r + d).$$

To be able to grasp a better understanding of Definition 5.34, see the examples below.

Example 5.37. Consider the $\mathbb{R}^{\mathbb{Z}_<}$-valued function $f(x)$ defined as follows:

$$f(\mathbf{x}) = \begin{cases} \mathbf{x} & \mathtt{St}(\mathbf{x}) \leq 1 \\ \mathbf{x} \widehat{+} 1 & \text{otherwise.} \end{cases}$$

First we need to understand clearly how this function actually works. Figure 5.4.4, where i denotes an arbitrary infinitesimal number, illustrates to us what the

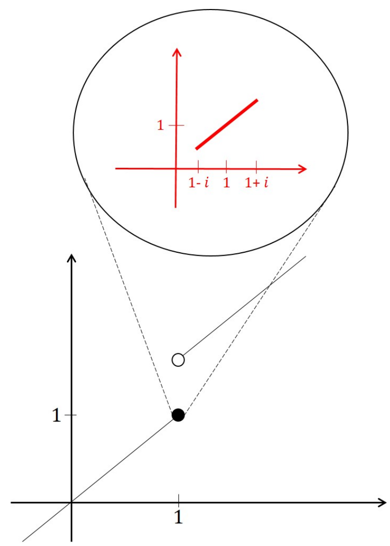


Figure 5.4.4: Illustration of Function $f(x)$ in Example 5.37

function $f(\mathbf{x})$ looks like. Notice that at $\mathbf{x} = 1$, what looks like a point in real numbers is actually a (constant) line when we zoom in deep enough into $\mathbb{R}^{\mathbb{Z}_<}$[11]. So how about the continuity of this function? It is obvious that $f(\mathbf{x})$ is not $0, 0$-continuous (by taking, for example, $\mathfrak{e}_{1_0} = \frac{1}{2}$ and $\mathbf{x} = 1.5$). However, interestingly enough, *it is* $0, 1$-*continuous* by taking $\mathfrak{e}_{2_1} \in \Delta^1$. Why is that? The fact that $\mathfrak{e}_{2_1} \in \Delta^1$ and that it has to depend on $\mathfrak{e}_{1_0}$ means that $\mathfrak{e}_{1_0}$ has to be in Δ^1 as well. Now, assigning $\mathfrak{e}_{2_1} = \mathfrak{e}_{1_0}$ is sufficient to prove its $0, 1$-continuity.

Example 5.38. The identity function $f(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$ is $0, 0$-continuous, just like in reals. However, it is *not* $1, 0$-continuous because for any point $\mathbf{c}$, there is an $\mathfrak{e}_{1_1} = \boldsymbol{\varepsilon}$ such that for every $\mathfrak{e}_{2_0} = r$ where $r \in \mathbb{R}$, $|\mathbf{x} \widehat{-} \mathbf{c}| < r$ but $f(\mathbf{x}) - f(c) \geq \boldsymbol{\varepsilon}$. In fact, identity function is k, n-continuous only when

[11]We have to be really careful here because if the first condition there is $\mathbf{x} \leq 1$ instead of $\mathtt{St}(\mathbf{x}) \leq 1$, then there will be no line there – it will be *exactly* one point.

$k \leq n$, but not otherwise.

The next theorem below is very interesting in as much as it enables us to classify whether a function is a constant function or not by using (k, n)-continuity.

Theorem 5.39. *For any function f, if f is (k, n)-continuous for any $k, n \in \mathbb{N} \cup \{0\}$, then f is a constant function.*

PROOF: Here we want to prove its contrapositive, in other words, if f is not constant, then there exist k, n such that f is not k, n-continuous. Because f is not constant, there will be a, b in the domain such that $f(a) \neq f(b)$ and suppose that $|f(a) - f(b)| \in \Delta^m$ such that $|a - b| \in \Delta^l$. By this construction, f will not be $(m, l - 1)$-continuous and so we can take $k = m$ and $n = l - 1$. ■

The theorem below is a generalisation of Theorem 5.39.

Theorem 5.40. *If there exists m for all k such that a function f is (k, m)-continuous, then f will be constant in Δ^m-neighbourhood.*

The next interesting question is: what is the relation between, for example, $0, 1$-continuity and $0, 2$-continuity? In general, what is the relation between k, n-continuity and $k, (n + 1)$-continuity? And also between k, n-continuity and $(k + 1), n$-continuity? See these two theorems below.

Theorem 5.41. *For any function f, if f is k, n-continuous, then f is also $k, (n + 1)$-continuous.*

PROOF: Suppose that a function f is k, n-continuous at point $\mathbf{c}$. This would mean that $\forall \mathfrak{e}_{1_k}$, $\exists \mathfrak{e}_{2_n}$ such that if $|\mathbf{x} \widehat{-} \mathbf{c}| < \mathfrak{e}_{2_n}$, then $|f(\mathbf{x}) \widehat{-} f(\mathbf{c})| < \mathfrak{e}_{1_k}$. By using the same $\mathfrak{e}_{1_k}$ and from Remark 5.36, we can surely find $\mathfrak{e}_{2_{(n+1)}} = \mathfrak{e}_{2_n} \widehat{\times} \boldsymbol{\varepsilon}$ such that for all $x \in (\mathbf{c} \widehat{-} \mathfrak{e}_{2_{(n+1)}}, \mathbf{c} \widehat{+} \mathfrak{e}_{2_{(n+1)}}), f(x) \in (f(\mathbf{c}) \widehat{-} \mathfrak{e}_{1_k}, f(\mathbf{c}) \widehat{+} \mathfrak{e}_{1_k})$. ∎

Example 5.42. By Theorem 5.41, the function $f(x)$ in Example 5.37 is also $0, 2$-continuous, $0, 3$-continuous and so on.

Theorem 5.43. *For any function f, if f is $(k+1), n$-continuous, then f is also k, n-continuous.*

PROOF: Suppose that a function f is $(k+1), n$-continuous at point $\mathbf{c}$ and the set Δ^m defined as in Theorem 5.41. The fact that f is $(k+1), n$-continuous means that $\forall \mathfrak{e}_{1_{(k+1)}}$, $\exists \mathfrak{e}_{2_n}$ such that if $|\mathbf{x} \widehat{-} \mathbf{c}| < \mathfrak{e}_{2_n}$, then $|f(\mathbf{x}) \widehat{-} f(\mathbf{c})| < \mathfrak{e}_{1_{(k+1)}}$ is hold. Here we want to prove that $\forall \mathfrak{e}_{1_k}$, $\exists \mathfrak{e}_{2_n}$ such that if $|\mathbf{x} \widehat{-} \mathbf{c}| < \mathfrak{e}_{2_n}$, then $|f(\mathbf{x}) \widehat{-} f(\mathbf{c})| < \mathfrak{e}_{1_k}$. This actually follows directly from Remark 5.36 as
$(f(\mathbf{c}) \widehat{-} \mathfrak{e}_{1_{(k+1)}}, f(\mathbf{c}) \widehat{+} \mathfrak{e}_{1_{(k+1)}}) \subseteq (f(\mathbf{c}) \widehat{-} \mathfrak{e}_{1_k}, f(\mathbf{c}) \widehat{+} \mathfrak{e}_{1_k})$. ∎

Now suppose that f and g are continuous functions in $\mathbb{R}^{\mathbb{Z}_<}$. We will examine how the arithmetic of those two continuous functions works. It is clear that (k, n)-continuity is closed under addition and subtraction, i.e. $f + g$ and $f - g$ are both continuous. The composition and multiplication of two continuous functions are particularly interesting. Look at the two following theorems carefully.

Theorem 5.44. *If f is a (k, n)-continuous function and g is an (n, q)-continuous function, then $f \circ g$ will be (k, q)-continuous.*

PROOF: Since f is (k, n)-continuous at $g(c)$, our definition of continuity tells us that for all $\mathfrak{e}_{1_k} > 0$, there exists $\mathfrak{e}_{2_n}$ such that

$$\text{if } |g(x) - g(a)| < \mathfrak{e}_{2_n}, \text{ then } |f(g(x)) - f(g(a))| < \mathfrak{e}_{1_k}.$$

Also since g is (n, q)-continuous at c, there exists $\mathfrak{e}_{2_q}$ such that

$$\text{if } |x - a| < \mathfrak{e}_{2_q}, \text{ then } |g(x) - g(a)| < \mathfrak{e}_{2_n}.$$

I have taken $\mathfrak{e}_{1_n} = \mathfrak{e}_{2_n}$ here. Now this tells us that for all $\mathfrak{e}_{1_k} > 0$, there exists $\mathfrak{e}_{2_q}$ (and an $\mathfrak{e}_{2_n}$) such that

$$\text{if } |x - a| < \mathfrak{e}_{2_q}, \text{ then } |g(x) - g(a)| < \mathfrak{e}_{2_n} \text{ which implies that}$$
$$|f(g(x)) - f(g(c))| < \mathfrak{e}_{1_k},$$

which is what we wanted to show. ∎

Theorem 5.45. *Suppose that f, g are finite-valued functions. If f is a (k, n)-continuous function and g is an (l, o)-continuous function, then the function $H = f \cdot g$ will be $(\max\{k, l\}, \min\{n, o\})$-continuous.*

PROOF: Let f, g be given such that f is (k, n)-continuous and g is (l, o)-continuous. Now let H be defined by $H(x) = f(x)g(x)$ and so we want to show that H is $(\max\{k, l\}, \min\{n, o\})$-continuous, that is, for all $c \in \mathbb{R}^{\mathbb{Z}_<}$, for every $\mathfrak{e}_{1_{\max\{k,l\}}} > 0$, there exists $\mathfrak{e}_{2_{\min\{n,o\}}} > 0$ such that for all $x \in \mathbb{R}^{\mathbb{Z}_<}$ with $|x - c| < \mathfrak{e}_{2_{\min\{n,o\}}}$, $|H(x) - H(c)| < \mathfrak{e}_{1_{\max\{k,l\}}}$ holds.

Now let c and $\mathfrak{e}_{1_{\max\{k,l\}}}$ be given and we choose $\mathfrak{e}_2$ such that $\mathfrak{e}_2 \in \Delta^{\min\{n,o\}}$, i.e. $\mathfrak{e}_2 = \mathfrak{e}_{2_{\min\{n,o\}}}$. Then for all $x \in \mathbb{R}^{\mathbb{Z}_<}$ with $|x - c| < \mathfrak{e}_{2_{\min\{n,o\}}}$,

$$|H(x) - H(c)| = |f(x)g(x) - f(c)g(c)|$$

$$
\begin{aligned}
&= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(c)g(c)| \\
&\leq |f(x)g(x) - f(c)g(c)| + |f(x)g(a) - f(c)g(c)| \\
&= |f(x)(g(x) - g(a))| + |g(a)(f(x)f(c))| \\
&< |f(x)|\mathfrak{e}_{1_l} + |g(a)|\mathfrak{e}_{1_k} && (5.8)
\end{aligned}
$$

Note that because f and g are limited-valued function, then $|f(x)|\mathfrak{e}_{1_l}$ and $|g(a)|\mathfrak{e}_{1_k}$ are still in Δ^l and Δ^k, respectively. This means that the right side of Inequality 5.8 will be in $\Delta^{\max\{k,l\}}$ and so $H(x) - H(c) < \mathfrak{e}_{1_{\max\{k,l\}}}$ is hold. ■

It is worth pointing out here that the definition of (k, n)-continuity is a much more fine-grained notion than the classical continuity. This is self-explaining by the use of those two variables k and n which makes us able to take much more infinitesimals – in other words, we will be able to examine a far greater depth – than in the classical definition. Furthermore, there might be some possible connections to one of the quantum phenomenons in physics: action at a distance. This concept is typically characterized in terms of some cause producing a spatially separated effect in the absence of any medium by which the causal interaction is transmitted [16] and closely connected to the question of what the deepest level of physical reality is [32, pg. 168]. Note that the researches on this phenomenon still being done up until now, as can be seen for example [33], [41] and [49].

5.4.1 Topological Continuity

Definition 5.46. A function f from a topological space (X, τ_1) to a topological space (Y, τ_2) is a function $f : X \to Y$.

From now on, we will abbreviate this function notation by $f : X \to Y$ or simply f every time the topologies in X and Y need not be explicitly mentioned. Also, f^{-1} denotes the inverse image of f as usual.

Definition 5.47 (St-continuous)**.** A function $f : X \to Y$ between topological spaces is *standard topologically continuous*, denoted by St-continuous, if

$$f^{-1}(U) \subseteq X \text{ is St-open whenever } U \subseteq Y \text{ is St-open.}$$

Definition 5.48 ($\mathfrak{e}$-continuous)**.** A function $f : X \to Y$ between topological spaces is *infinitesimally topologically continuous*, denoted by $\mathfrak{e}$-continuous, if

$$f^{-1}(U) \subseteq X \text{ is } \mathfrak{e}\text{-open whenever } U \subseteq Y \text{ is } \mathfrak{e}\text{-open.}$$

Theorem 5.49. *Suppose that $X, Y \subseteq \mathbb{R}^{\mathbb{Z}_<}$. Under the metric d, a function $f : X \to Y$ is St-continuous if and only if f satisfies $\mathrm{ED}_{\mathrm{CLASS}}$ definition (Definition 5.15).*

Proof: We need to prove the implication both ways.

1. We want to prove that if f is St-continuous, then f satisfies $\mathrm{ED}_{\mathrm{CLASS}}$. Suppose that f is St-continuous and let $\mathbf{x_0} \in X$ and $n \in \mathbb{N} > 0$. Then, the ball

$$B_{f(\mathbf{x}_0)}(1/n) = \{\mathbf{y} \in Y \mid \mathsf{d}(\mathbf{y}, f(\mathbf{x}_0)) < 1/n\}$$

is open in Y, and hence $f^{-1}(B_{f(\mathbf{x}_0)})$ is open in X. Since $\mathbf{x}_0 \in f^{-1}(B_{f(\mathbf{x}_0)})$, there exists some balls of radius $1/m$ for some $m \in \mathbb{N}$ such that

$$B_{\mathbf{x}_0}(1/m) \subseteq f^{-1}(B_{f(\mathbf{x}_0)}).$$

This is exactly what the $\mathrm{ED}_{\mathrm{CLASS}}$ says.

2. We want to prove that if f satisfies $\mathrm{ED}_{\mathrm{CLASS}}$, then f is St-continuous.

   Suppose that f satisfies $\mathrm{ED}_{\mathrm{CLASS}}$ and let $U \subseteq Y$ is open. By Definition 4.6, for all $\mathbf{y} \in U$ there exists some $d_y = 1/n_y$ where $n_y \in \mathbb{N}$ such that

$$B_{\mathbf{y}}(d_y) \subseteq U$$

   and in fact,

$$U = \bigcup_{y \in U} B_{d_y}(\mathbf{y}). \tag{5.9}$$

   Now we claim that $f^{-1}(U)$ is open in X and suppose that $\mathbf{x}_0 \in f^{-1}(U)$. Then $f(\mathbf{x}_0) \in U$ and so from Equation 5.9, $f(\mathbf{x}_0) \in B_{d_{y_0}}(\mathbf{y}_0)$ for some $\mathbf{y}_0 \in U$ and $d_{y_0} = 1/n_{y_0}$ for some $n_{y_0} \in \mathbb{N}$, i.e. $\mathrm{d}(f(\mathbf{x}_0), \mathbf{y}_0) < d_{y_0}$. Now define

$$e = d_{y_0} - \mathrm{d}(f(\mathbf{x}_0), \mathbf{y}_0) > 0. \tag{5.10}$$

   By Definition 5.15, there exists some $m \in \mathbb{N}$ such that

$$\text{if } \mathbf{x} \in X \text{ and } \mathrm{d}(\mathbf{x}, \mathbf{x}_0) < 1/m, \text{ then } \mathrm{d}(f(\mathbf{x}), f(\mathbf{x}_0)) < e. \tag{5.11}$$

Now we claim that

$$B_{\mathbf{x}_0}(1/m) \subseteq f^{-1}(U), \tag{5.12}$$

which will actually show that $f^{-1}(U)$ is indeed open. To this end, let $x \in B_{\mathbf{x}_0}(1/m)$, i.e. $\mathtt{d}(\mathbf{x}, \mathbf{x}_0) < 1/m$. Then from (5.11), we have $\mathtt{d}(f(\mathbf{x}), f(\mathbf{x}_0)) < e$. Then, the triangle inequality and (5.10) imply that

$$\mathtt{d}(f(\mathbf{x}), \mathbf{y}_0) \leq \mathtt{d}(f(\mathbf{x}), f(\mathbf{x}_0)) + \mathtt{d}(f(\mathbf{x}_0), \mathbf{y}_0) < e + \mathtt{d}_\psi(f(\mathbf{x}_0), \mathbf{y}_0) = d_{y_0}.$$

This means that $f(x) \in B_{\mathbf{y}_0}(d_{y_0}) \subseteq U$, so that $x \in f^{-1}(U)$. Therefore, (5.12) holds, as claimed.

And so from those two points above, we have proved what we want. ■

Theorem 5.50. *Suppose that $X, Y \subseteq \mathbb{R}^{\mathbb{Z}_<}$. Under the metric* $\mathtt{d}$*, a function $f : X \to Y$ is $\mathfrak{e}$-continuous if and only if f satisfies* ED *definition.*

Proof: The proof of this theorem is similar with the one in Theorem 5.49 with some slight modifications in the distances (from $1/n$ for some $n \in \mathbb{N}$ into $\mathfrak{e} \in \mathbb{R}^{\mathbb{Z}_<}$). ■

5.5 Convergence

When we are talking about sequences, it is necessary to talk also about what it means when we say that a sequence is convergent to a particular number. In this section we will present not only some possible definitions that can be used to define convergence in $\mathbb{R}^{\mathbb{Z}_<}$, but also the problems which occur when we apply these definitions to $\mathbb{R}^{\mathbb{Z}_<}$.

Definition 5.51 (Classical Convergence)**.** A sequence s_n converges to s iff,

$$\forall m \in \mathbb{N}, \exists N \text{ such that } \forall n > N, |s_n - s| < \tfrac{1}{m}.$$

We write $\mathrm{CC}(s_n, s)$ to denote that a sequence s_n is classically convergent to s.

Definition 5.51 above is the standard definition of how we define the notion of convergent classically.

Definition 5.52 (Hyperconvergence)**.** A sequence s_n converges to s iff,

$$\forall r > 0, \exists N \text{ such that } \forall n > N, |s_n - s| < r.$$

We write $\mathrm{HC}(s_n, s)$ to denote that a sequence s_n is hyperconvergent to s. The interpretation of r can be either in $\mathbb{R}$ or in $\mathbb{R}^{\mathbb{Z}_<}$.

Example 5.53. Suppose that we have a sequence $s_n = \varepsilon^n$ as follows:

$$\begin{gathered} S_1 = \varepsilon = \langle \widehat{0}, 1, 0, \ldots \rangle \\ S_2 = \varepsilon^2 = \langle \widehat{0}, 0, 1, 0, \ldots \rangle \\ S_3 = \varepsilon^3 = \langle \widehat{0}, 0, 0, 1, 0, \ldots \rangle \\ \vdots \end{gathered}$$

This sequence s_n will hyperconverge to $\langle \widehat{0}, 0, 0, \ldots \rangle$, i.e. s_n satisfies $\mathrm{HC}(s_n, \mathbf{0})$.

Theorem 5.54. *For any sequence s_n, $\mathbb{R} \models \mathrm{CC}(s_n, s) \leftrightarrow \mathrm{HC}(s_n, s)$.*

PROOF: The proof from HC to CC is obvious. Now suppose that a sequence s_n satisfies $\mathrm{CC}(s_n)$ and w.l.o.g. we assume that the r in HC definition is between 0 and 1. From the Archimedean property of reals we know that for every $0 < r < 1$, we can find an $m \in \mathbb{N}$ such that $\frac{1}{m} < r$, and so because of $\mathrm{CC}(s_n)$, we have $|s_n - s| < \frac{1}{m} < r$. ■

Theorem 5.55. *For any sequence s_n in $\mathbb{R}^{\mathbb{Z}_<}$, $\mathrm{HC}(s_n, s)$ always implies $\mathrm{CC}(s_n, s)$. However, there exists a sequence (t_n) such that*

$$\mathbb{R}^{\mathbb{Z}_<} \models \mathrm{CC}(t_n, s) \not\rightarrow \mathrm{HC}(t_n, s).$$

PROOF:

1. To prove the first clause, suppose that a sequence s_n satisfies $\mathrm{HC}(s_n, s)$. This means that we are able to find a number N such that $\forall n \geq N$, $|s_n - s| < r$ for any $r \in \mathbb{R}^{\mathbb{Z}_<}$ which includes infinitesimals. By using the same N, s_n will satisfy $\mathrm{CC}(s_n, s)$.

2. To prove the second clause, take the sequence $t_n = \frac{1}{n}$ where $n \in \mathbb{N}$. This sequence satisfies $\mathrm{CC}(t_n, 0)$, but it does not satisfy $\mathrm{HC}(t_n, s)$ for any s (as any $r \in \Delta$ will satisfy the negation of Definition 5.52).

■

Lemma 5.56. *Let (s_n) be a sequence in $\mathbb{R}^{\mathbb{Z}_<}$ such that HC(s_n, s) is hold. Then, HC($|s_n|, |s|$) is hold.*

PROOF: Let $r > 0 \in \mathbb{R}^{\mathbb{Z}<}$ be given. Then this means that there exists $N \in \mathbb{N}$ such that $\forall m > N$, $|s_m - s| < r$. Therefore, we also have

$$\forall m > N, ||s_m| - |s|| \leq |s_m - s| < r.$$

Hence, HC$(|s_n|, |s|)$ is true. ■

Note that the converse of Lemma 5.56 is not necessarily true.

Theorem 5.57. *Let $X \subset \mathbb{R}^{\mathbb{Z}<}$ and $f : X \to \mathbb{R}^{\mathbb{Z}<}$. Then f is $\mathfrak{e}$-continuous at $x_0 \in X$ iff for any sequence x_n in X that satisfies $HC(x_n, x_0)$, the sequence $f(x_n)$ satisfies $HC(f(x_n), f(x_0))$.*

PROOF: Suppose that f is $\mathfrak{e}$-continuous at x_0 and let the sequence x_n be defined in X and that x_n hyper converges to x_0. Now let $\mathfrak{e} > 0$ be given. Then from Theorem 5.50, there exists $\mathfrak{e}_2 > 0 \in \mathbb{R}^{\mathbb{Z}<}$ such that

$$\text{if } x \in X \text{ and } |x - x_0| < \mathfrak{e}_2, \text{ then } |f(x) - f(x_0)| < \mathfrak{e}.$$

Now since x_n hyper converges to x_0, then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ $|x_n - x_0| < \mathfrak{e}_2$. Thus we have

$$\forall n \geq N \, |f(x_n) - f(x_0)| < \mathfrak{e}.$$

and so the sequence $f(x_n)$ hyper converges to $f(x_0)$.

For the converse, we will prove the contrapositive. Suppose that f is not $\mathfrak{e}$-continuous at x_0. Then it means that there exists $\mathfrak{e}_0 > 0 \in \mathbb{R}^{\mathbb{Z}<}$ such that for all $\mathfrak{e}_2 > 0 \in \mathbb{R}^{\mathbb{Z}<}$, there exists $x \in X$ such that $|x - x_0| < \mathfrak{e}_2$ but

$|f(x) - f(x_0)| > \mathfrak{e}_0$. In particular, for all $n \in \mathbb{N}$, there exists $x_n \in X$ such that $|x_n - x_0| < \mathfrak{e}_2$ and $|f(x_n) - f(x_0)| > \mathfrak{e}_0$. Thus x_n is a sequence in X that hyper converges to x_0, but the sequence $f(x_n)$ does not hyper converge to $f(x_0)$. ∎

Definition 5.58. Let s_n be a sequence in $\mathbb{R}^{\mathbb{Z}<}$. Then we say that s_n is a hyper Cauchy sequence iff $\forall \mathfrak{e} \in \mathbb{R}^{\mathbb{Z}<}$, $\exists N \in \mathbb{N}$ such that

$$\forall l, m \geq N \, |s_l - s_m| < \mathfrak{e}.$$

Theorem 5.59. *Every hyper convergent sequence in $\mathbb{R}^{\mathbb{Z}<}$ is a hyper Cauchy sequence.*

PROOF: Let s_n be a sequence in $\mathbb{R}^{\mathbb{Z}<}$ that satisfies $\mathrm{HC}(s_n, s)$. We want to show that s_n is hyper Cauchy. Let $\mathfrak{e} \in \mathbb{R}^{\mathbb{Z}<}$ be given. Then there exists $N \in \mathbb{N}$ such that $\forall n > N$, $|s_n - s| < \frac{\mathfrak{e}}{2}$. Then for all $l, m > N$, we have

$$|s_l - s_m| = |s_l - s - (s_m - s)| \leq |s_l - s| + |s_m - s| < \frac{\mathfrak{e}}{2} + \frac{\mathfrak{e}}{2} = \mathfrak{e}$$

and so s_n is hyper Cauchy. ∎

Theorem 5.60. *The set $\mathbb{R}^{\mathbb{Z}<}$ is hyper Cauchy complete with respect to the $\mathfrak{e}$-topology.*

Lemma 5.61. *Let s_n be a sequence in $\mathbb{R}^{\mathbb{Z}<}$ whose members are just real numbers – that is, for all $s \in s_n$, $Nst_\varepsilon(s) = Nst_\omega(s) = \emptyset$. Then s_n is hyper Cauchy if and only if there exists $N \in \mathbb{N}$ such that $s_m = s_N$ for all $m \geq N$.*

PROOF: Let s_n be a hyper Cauchy sequence in $\mathbb{R}^{\mathbb{Z}<}$ whose members are real numbers. Then there exists $N \in \mathbb{N}$ such that

$$|s_m - s_l| < \varepsilon \text{ for all } m, l \geq N. \tag{5.13}$$

Since s_n is a sequence of real numbers, we obtain from Inequality 5.13 that for all $m, l \geq N$, $|s_m - s_l| = 0$ and so $s_m = s_N$ for all $m \geq N$.

Conversely, let s_n be a sequence in $\mathbb{R}^{\mathbb{Z}_<}$ whose members are real numbers and assume that there exists $N \in \mathbb{N}$ such that $s_m = s_N$ for all $m \geq N$. Now let $\mathfrak{e} > 0$ be given. We have that for all $l, m \geq N$, $|s_m - s_l| = 0 < \mathfrak{e}$ and so s_n is hyper Cauchy. ■

Another possible way to define convergence in our set is through the concept of ℓ^∞ as follows:

Definition 5.62 ($\mathbb{R}^{\mathbb{Z}_<}$-Convergence)**.** Suppose that s_n is a sequence where every member of it is another sequence itself, i.e.

$$s_n = (s_n)_1, (s_n)_2, (s_n)_3, \ldots, (s_n)_i, \ldots.$$

Then, s_n converges to $\mathbf{s}$ iff $\forall m \in \mathbb{N}$, $\exists N$ such that

$$\forall n \geq N, \forall i \, |(s_n)_i - s_i| < \tfrac{1}{m}.$$

We write $\mathrm{RC}(s_n, s)$ to denote that a sequence s_n is $\mathbb{R}^{\mathbb{Z}_<}$-convergent to s.

Example 5.63. The sequence $s_n = \langle \widehat{\tfrac{1}{n}}, 0, 0, \ldots \rangle$ is $\mathbb{R}^{\mathbb{Z}_<}$-convergent to $\mathbf{0}$.

The next interesting question is which of the three definitions above can be used to define convergence in $\mathbb{R}^{\mathbb{Z}_<}$? Unfortunately, neither of them is adequate to serve as *the* definition of convergence in our set. The three examples below demonstrate why. The first shows that when Classical Convergence is adopted in $\mathbb{R}^{\mathbb{Z}_<}$, convergence is no longer unique, the second shows how as a result of adoption Definition 5.52 something unexpected occurs in our set, the last why $\mathbb{R}^{\mathbb{Z}_<}$-convergence is not adequate.

Example 5.64. Suppose that s_n is a sequence defined by:

$$s_n = \langle \widehat{0}, n, 0, \ldots \rangle.$$

Then by using Definition 5.51 above and the fact that any infinitesimals are less than any rational numbers, s_n classically converges to 100ε, 200ε, 300ε, and so on. In other words, the sequence s_n satisfies $(\mathrm{CC}(s_n, 100\varepsilon))$, $(\mathrm{CC}(s_n, 200\varepsilon))$, $(\mathrm{CC}(s_n, 300\varepsilon))$, and so on.

Example 5.65. Using Definition 5.52, the sequence $s_n = \langle \widehat{\frac{1}{n}}, 0, 0, \ldots \rangle$ does not converge in the usual sense to 0, i.e. s_n does not satisfy $\mathrm{HC}(s_n, \mathbf{0})$. Taking $r = \varepsilon = \langle \widehat{0}, 1, 0, \ldots \rangle$ and $n = N + 1$ will show this.

Example 5.66. The sequence $s_n = \varepsilon^n$ does not $\mathbb{R}^{\mathbb{Z}_<}$-converge to $\mathbf{0}$, as it should do intuitively.

Thus, this leaves us with the three definitions of convergence used in $\mathbb{R}^{\mathbb{Z}_<}$. There is no one definition of convergence in our set. This is not necessarily a bad thing, it simply means that our notion of convergence will differ from that of classical analysis. Figure 5.5.1 shows the connection between the three definitions in $\mathbb{R}^{\mathbb{Z}_<}$.

Note that our attempts to have a proper notion of continuity and convergence in $\mathbb{R}^{\mathbb{Z}_<}$ can be used in the area of reverse mathematics. From what we have done here, it can help us to gain a better understanding about some *necessary condition*, for example, for a function f to be continuous or for a sequence to be convergent.

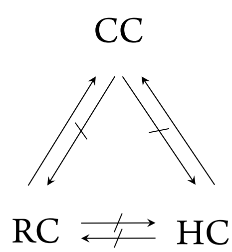


Figure 5.5.1: Relationship among CC, RC, and HC in $\mathbb{R}^{\mathbb{Z}_<}$

"Given the pace of technology, I propose that we leave math to the machines and go play outside""

Bill Watterson in *Calvin and Hobbes*

6
Computability in $\mathbb{R}^{\mathbb{Z}<}$

Before discussing computable numbers, we first need to define what we mean by a function being computable. Informally speaking, a computable function is a function *f* which could, in principle, be calculated using a mechanical calculation tool and given a finite amount of time. In the language of computer science, we would say that there is an algorithm computing the function. A computable real number is, in essence, a number whose approximations are given by a computable function.

The notion of a function $\mathbb{N} \to \mathbb{N}$ being computable is well understood. In fact all definitions so far capturing this idea, such as Turing Machines, Markov Algorithms, Lambda Calculus, the (partial) recursive functions, and many more have all led to the same class of functions. This, in turn, has led to the so called Church-Markov-Turing thesis, which says that this class *is* exactly what computable intuitively means. Given computable pairing functions this also, immediately, leads to a notion of computability for other function types such as $\mathbb{N}^k \to \mathbb{N}^m$, $\mathbb{N} \to \mathbb{Z}$ or $\mathbb{N} \to \mathbb{Q}$. If we see a real number as a sequence of rational approximations, we also get a definition of a computable real number.

However, we do have to be a bit careful. There are many equivalent formulations for when a real number r is computable that work well in practice, such as if

- there is a finite machine that computes a quickly converging[12] Cauchy sequence that converges to r, or
- it can be approximated by some computable function $f : \mathbb{N} \to \mathbb{Z}$ such that: given any positive integer n, the function produces an integer $f(n)$ such that

$$\frac{f(n)-1}{n} \leq a \leq \frac{f(n)+1}{n}.$$

We denote the set of all computable real numbers by $\mathbb{R}_c$. It is well known (and

[12] That is with a fixed modulus of Cauchyness.

also well studied) that many real numbers such as π or e are computable. However, not every real number is computable.

One possibility that does not turn out to be useful is to write down a real number by using the decimal representation. Let us consider numbers r, such that there is an algorithm whose input is n and it will give the n^{th}-digit of its decimal representation. The set of all real numbers that have a computable decimal representation is denoted by $\mathbb{R}_d$.

Remark 6.1. Although the set $\mathbb{R}_d$ is closed under the usual arithmetic operations, we have to be careful of what it really means. Take, for example, addition. We know that if x and y are in $\mathbb{R}_d$, then $x + y$ is in $\mathbb{R}_d$ as well. However, it does *not* mean that the addition operator itself is computable.

These ideas of computability can be extended to infinitesimals!

Now in $\mathbb{R}^{\mathbb{Z}_<}$, we define its member to be computable if it satisfies the condition as stated in Definition 6.2.

Definition 6.2. A number $\mathbf{z} \in \mathbb{R}^{\mathbb{Z}_<}$ is computable iff there is a computable function f such that $f(n, \cdot)$ are computable numbers and

$$\mathbf{z} = \langle f(1, \cdot), f(2, \cdot), \ldots, \widehat{f(l, \cdot)}, \ldots \rangle$$

where $l = f(0, 0)$ denotes the index where the $\mathtt{St}(\mathbf{z})$ is. We denote the set of all computable members of $\mathbb{R}^{\mathbb{Z}_<}$ by $\mathbb{R}_c^{Z_<}$.

In this chapter, we are going to show that the standard arithmetic operations (functions) in $\mathbb{R}^{\mathbb{Z}_<}$ are computable (provided that the domain and codomain of those functions are $\mathbb{R}_c^{Z<}$). This will be done by explicitly showing the program for each one of them. We will actually use a concrete implementation of these ideas in the programming language Python, whose syntax should be intuitively understandable even by those not familiar with it. We will also not show that our programs are correct, since they are so short that such a proof would be trivial.

Assuming that we already have a working implementation of $\mathbb{R}_c$, our class $\mathbb{R}_c^{Z<}$ can be implemented as in Listing A.1, where we define the members of our set $\mathbb{R}_c^{Z<}$ (basically just a container for the index l as above and the sequence of digits) and how the string representation will look like.

```
class infreal:
  def __init__(self, digits, k=0):
    self.k = k
    self.digits = digits
  def __repr__(self):
    if self.k == 0:
      return "^" + ", ".join([str(self.digits(i)) for i in range(self.k,self.k+7)]) + ", ..."
    else:
      return ", ".join([str(self.digits(i)) for i in range(self.k)]) + ", ^" + ", ".join([str(self.
     digits(i)) for i in range(self.k,self.k+7)]) + ", ..."
  def __getitem__(self, key): return self.digits(key)
```

Listing 6.1: How to define the members of $\mathbb{R}_c^{Z<}$.

Example 6.3. Suppose that we want to write the number $\widehat{1}$. Then by writing `One=infreal(lambda n:one if n==0 else zero, 0)` (where `zero` and `one` are the real numbers 0 and 1, respectively), we have just created the number 1 in our system. The second argument of the function `infreal` there is just to give how many digits we want to have before the real part of our number (the number with a hat'). Its input and output will look like as follows:

```
>>> zero = real(0)
>>> one = real(1)
```

```
>>> One = infreal(lambda n: one if n==0 else zero, 0)
>>> One
    ^1, 0, 0, 0, 0, 0, 0, ...
```

Furthermore, we will also be able to know what is its n^{th} digit for any $n \in \mathbb{N}$. See the code below:

```
>>> One
    ^1, 0, 0, 0, 0, 0, 0, ...
>>> One[0]
    1
>>> One[-56]
    0
>>> One[2454]
    0
```

Example 6.4. Using the same way as the one in Example 6.3, we can define the number $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ in our system.

```
>>> Epsilon = infreal(lambda n: one if n == 1 else zero, 0)
>>> Epsilon
    ^0, 1, 0, 0, 0, 0, 0, ...
>>> Omega = infreal(lambda n: one if n == 0 else zero, 1)
>>> Omega
    1, ^0, 0, 0, 0, 0, 0, 0, ...
```

Also, we will be able to have exotic numbers such as $\mathrm{e} + 2\mathrm{e}\boldsymbol{\varepsilon} + 3\mathrm{e}\boldsymbol{\varepsilon}^2 + 4\mathrm{e}\boldsymbol{\varepsilon}^3 + \ldots$ and its code will be as follows:

```
>>> e = exp(rational(1,1))
>>> Funny = infreal(lambda n: real(rational(n + 1, 1)) * e if n > -1 else zero, 0)
>>> Funny
    ^2.71828, 5.43656, 8.15485, 10.8731, 13.5914, 16.3097, 19.028, ...
>>> Funny[43532]
    118335
>>> Funny[-12964]
    0
```

Theorems 6.5-6.9 show that addition, subtraction, and multiplication in $\mathbb{R}_c^{Z<}$ are computable.

Theorem 6.5. *Suppose that we have $x, y \in \mathbb{R}_c^{Z<}$. Then the function $\widehat{+}_c$ defined by*

$$\begin{aligned} \widehat{+}_c : \quad \mathbb{R}_c^{Z<} &\to \mathbb{R}_c^{Z<} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \mathbf{x} \widehat{+} \mathbf{y} \end{aligned}$$

is computable.

PROOF: The following code shows that the function $\widehat{+}_c$ defined above is computable.

```
def __add__(self, other):
  k = max(self.k, other.k)
  return infreal(lambda n: self.digits(n - (k-self.k)) + other.digits(n - (k-other.k)), k)
```

■

Example 6.6. Suppose that we want to add $\boldsymbol{\varepsilon}$, $\boldsymbol{\omega}$, and 1. Then we will have:

```
>>> Omega
1, ^0, 0, 0, 0, 0, 0, 0, ...
>>> Epsilon
^0, 1, 0, 0, 0, 0, 0, ...
>>> Epsilon + Omega + One
1, ^1, 1, 0, 0, 0, 0, 0, ...
```

Theorem 6.7. *Suppose that we have $x, y \in \mathbb{R}_c^{Z<}$. Then the function $\widehat{-}_c$ defined by*

$$\begin{aligned} \widehat{-}_c : \quad \mathbb{R}_c^{Z<} &\to \mathbb{R}_c^{Z<} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \mathbf{x} \widehat{-} \mathbf{y} \end{aligned}$$

is computable.

PROOF: The following code shows that the function $\widehat{-}_c$ defined above is computable.

```
47 def __neg__(self): return infreal(lambda n: -self[n], self.k)
48 def __sub__(self, other): return (self + (-other))
```

The definition in line 47 shows that the additive inverse function is computable.

∎

Example 6.8. Suppose that we want to add $\boldsymbol{\varepsilon} \widehat{-} \boldsymbol{\omega}$ to 1. Then we will have:

```
>>> Epsilon - Omega + One
    -1, ^1, 1, 0, 0, 0, 0, 0, ...
```

Theorem 6.9. *Suppose that we have $x, y \in \mathbb{R}_c^{Z<}$. Then the function $\widehat{\times}_c$ defined by*

$$\begin{aligned} \widehat{\times}_c : \quad \mathbb{R}_c^{Z<} &\to \mathbb{R}_c^{Z<} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \mathbf{x} \widehat{\times} \mathbf{y} \end{aligned}$$

is computable.

Proof: The following code shows that the function $\widehat{\times}_c$ defined above is computable.

```
def __mul__(self, other):
  k = self.k + other.k
  def digits(n):
    if n < 0:
      return zero
    else:
      return reduce((lambda x,y:x+y), [self.digits(i) * other.digits(n - i) for i in range(n + 1)
    ])
  return infreal(digits, k)
```

∎

Example 6.10. Suppose that we want to add $\boldsymbol{\varepsilon} \widehat{\times} \boldsymbol{\omega}$ to 1 and also $-\boldsymbol{\varepsilon}^2$ to 1. Then we will have:

```
>>> Epsilon * Omega + One
    0, ^2, 0, 0, 0, 0, 0, 0, ...
>>> Epsilon * -Epsilon + One
    ^1, 0, -1, 0, 0, 0, 0, ...
```

6.1 Some Remarks on Non-Computability in $\mathbb{R}^{\mathbb{Z}_<}$

Remark 6.11. Even though division on $\mathbb{R}_c$ is computable (assuming the input does not equal 0), the same can not be said of $\mathbb{R}_c^{\mathbb{Z}_<}$.

Remark 6.12. Suppose that we have a number $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_<}$. Then without any further information, the process of finding $\mathbf{x}^{-1}$ (the multiplicative inverse of $\mathbf{x}$) is not computable. One extra information needed to make it computable in $\mathbb{R}^{\mathbb{Z}_<}$ is how many digits we want to have in $\mathbf{x}^{-1}$, which of course will affect the accuracy of our result. More precisely, it is known that it is not possible to give an algorithm that, a given number $a \in \mathbb{R}_c$, decides whether $a = 0$ or $\neg a = 0$. So let $a \in \mathbb{R}_c$ and consider $\mathbf{z} = a + \boldsymbol{\varepsilon}$. We have $\boldsymbol{\varepsilon} \neq 0$. If $a = 0$ then $\mathbf{z}^{-1} = \boldsymbol{\omega}$. If $a \neq 0$ then $z^{-1} < \boldsymbol{\omega}$. Thus by checking whether the ω-part of z^{-1} is less than 1 or greater than 0, we would be able to decide whether $a = 0$ or $\neg a = 0$.

Remark 6.13. Similarly surprising, we can show that the absolute value function, which is computable for $\mathbb{R}_c$, is not computable for $\mathbb{R}_c^{\mathbb{Z}_<}$. Here the absolute value function is the function

$$|\mathbf{z}| = \begin{cases} \mathbf{z} & \text{if } \mathbf{z} \geq 0 \\ -\mathbf{z} & \text{if } \mathbf{z} < 0 \end{cases}$$

Similar to the above, take $a \in \mathbb{R}_c$ and consider $z = |a| - \boldsymbol{\varepsilon}$. If $a = 0$ then $|z| = -|a| + \boldsymbol{\varepsilon}$. If $a \neq 0$ then $|z| = |a| - \boldsymbol{\varepsilon}$. Thus by checking the ε-part of $|z|$, we would be able to decide whether $a = 0$ or $\neg a = 0$.

This also leads to comparison not being computable. Now this is also the case in $\mathbb{R}_c$. However, for numbers $x, y \in \mathbb{R}_c$ such that $x \neq y$, we can decide whether $x < y$ or $x > y$. This does not extend to $\mathbb{R}_c^{Z<}$:

Remark 6.14. Comparison among the members in $\mathbb{R}_c^{Z<}$ is not computable. Again, let $a \in \mathbb{R}_c$ and consider $\mathbf{x} = |a| + \boldsymbol{\varepsilon}$ and $\mathbf{y} = 2|a|$. If $a = 0$ then $\mathbf{x} > \mathbf{y}$, and if $a \neq 0$ then $\mathbf{x} < \mathbf{y}$. Thus, once again we would be able to decide whether $a = 0$ or $\neg a = 0$.

"You have declared that your goal is to reach the end of infinity, i.e. you have stated that there is not any goal. The measure of your success is not the distance to the finish but the distance from the start... but if you were able to see the whole labyrinth from above..."

Arkadi N and Boris N. Strugazky, *Far Rainbow*

7

Conclusion

Inconsistencies in mathematics often relate to the existence of infinitesimals and infinities. Nevertheless, these quantities are still used until today as can be seen in [43], and something we intend to maintain. One of the mathematical sets which contains infinitesimals and infinities are the hyperreals. One interesting question to ponder, however, is how this set relates to the set of the usual real numbers. Here the transfer principle gives the answer. However, as discussed in Chapter 2, there are some problems with the transfer principle, which also can be seen in [44], notably its non-computability and its technical difficulty in using it. Solving these problem is one of the motivations of this research. As it is clear from its name, the transfer principle exists due to the use of two languages in our theory. We solved this issue of transfer principle by collapsing those two languages into one language. However, a new problem arise: contradiction.

This thesis has proposed two ideas to solve this issue. On the one hand, we can change the logic used to one of the paraconsistent logics. On the other, we can have a subsystem in our language. The thesis favours the second idea due to reasons mentioned in the Chapter 1.

The basic idea of the second proposal is the Chunk and Permeate strategy, which was first introduced by Brown and Priest as can be seen in [8]. In this thesis, we invent a new set of numbers, $\widehat{\mathbb{R}}$ which will include infinities and infinitesimals in it and later on, due to the inconsistency found in that set, we divide it into two chunks. This leads us to the discovery of another new set $\mathbb{R}^{\mathbb{Z}_<}$ while proving the consistency of one of the chunks. This discovery is first discussed in Chapter 3. On the same chapter, we have also compared our new set with the recent existing theory, Grossone theory, and how our set can serve to prove the consistency of that latter theory.

In Chapter 4, we have discussed some of the topological aspects of $\mathbb{R}^{\mathbb{Z}_<}$, e.g. metrics in $\mathbb{R}^{\mathbb{Z}_<}$, balls, open sets, and etc. One of the interesting concepts that we introduced here is the different types of balls in $\mathbb{R}^{\mathbb{Z}_<}$, i.e. St-balls and $\mathfrak{e}$-balls (which are the infinitesimally small balls). Because of that, we have two different notions of openness and also of distinguishable points. Moreover, we also discuss about topological space and how $\mathbb{R}^{\mathbb{Z}_<}$ does not form a Hausdorff space under one topology and but it does under the other topology.

In Chapter 5, we develop some features of the set $\mathbb{R}^{\mathbb{Z}_<}$ more deeply. This includes a discussion of the derivative, integration, and also the notion of continuity and convergence. On the derivative side, we successfully develop a permeability relation such that the derivative function in $\mathbb{R}^{\mathbb{Z}_<}$ can be permeated to $\mathbb{R}$. Moreover, we also discuss some concepts of transcendental functions in $\mathbb{R}^{\mathbb{Z}_<}$, which are defined in term of series. We also show that the derivative of those functions can be permeated as well.

For the concept of continuity and convergence, there are some new notions introduced here. First of all, we discuss three different possible notions of continuity that can be applied to either the set $\mathbb{R}$ or $\mathbb{R}^{\mathbb{Z}_<}$. We also determine how they relate to each other in their respective model. While doing that, we define a new kind of fractals— infinitesimal fractals. After analysing three possible notions of continuity, we decided that the best notion that can be used in our setting $\mathbb{R}^{\mathbb{Z}_<}$ is the ε-δ definition and by doing that, we do not only preserve much of the spirit of classical analysis but also retain the intuition of infinitesimals. After establishing our position, we also introduce a more detailed notion of continuity which is called (k, n)-continuity where $k, n \in \mathbb{N} \cup \{0\}$ (as can be seen in Definition 5.34). We do some explorations on how this new notion of continuity behaves e.g. what happens with the composition of two continuous functions and also how this notion behaves under multiplication. It is worth pointing out here that this new notion of continuity is a much more fine-grained notion than the classical continuity. This leads to the discussion of topological aspects. To capture the idea of convergence, we proposed three notion that can be used in our set $\mathbb{R}^{\mathbb{Z}_<}$. In the same chapter, we argued that we cannot have **the** definition of convergence and that this is not necessarily a bad thing as it simply means that we have different notion of convergence than the one in classical analysis.

The last chapter of this thesis discusses about the computability aspect of our set $\mathbb{R}^{\mathbb{Z}_<}$. We succeeded in building a program, in Python, to show that we can have a computable number $\mathbb{R}^{\mathbb{Z}_<}$. The set of all these computable numbers is denoted by $\mathbb{R}_c^{\mathbb{Z}_<}$. In the last section, we show some interesting remarks regarding this computability issue.

In term of further research, below we indicate some possible areas of further development.

1. One can try to do infinitesimal analysis using the relevant logic **R**. By doing this, one can also compare what he might get with what we have done in this research, especially in term of usefulness and simplicity.

2. Regarding the transfer principle, our intuition says that it is equivalent to the notion of permeability in the chunk & permeate strategy. One can try to formally prove it or even disprove it.

3. In term of computability issue, using the calculus on $\mathbb{R}^{\mathbb{Z}_<}$, one can try to

formulate a necessary and sufficient condition for the derivatives of functions, for example, on a computer to exist. And perhaps, showing how to find these derivatives whenever they exist. This, of course, can also be applied to the other notions.

4. As we said in the previous chapters, some results in this research can help us to gain a better understanding in another area of research. The two that we mentioned before (in Chapter 5) are reverse mathematics and quantum physics. One can try to work more detail on this.

In general, with the new consistent set (which includes infinities and infinitesimals) that have been created in this work, new opportunities awaits mathematicians. One of the joys of mathematics is to explore a world which has no physical substance, and yet is everywhere in every aspect of our lives. Infinities and infinitesimals offer ways to explore hitherto unseen aspects of our world and our universe, by giving us the vision to see the greatest and smallest aspects of life. Even a naïve set, when it demonstrates harmony, offer another dimension of even clearer precision.

A

Code for Computable $\mathbb{R}^{\mathbb{Z}<}$

```
from functools import reduce

def euclid(a, b):
while b != 0:
a, b = b, a % b
return a

def factorial(n):
if n == 1 or n == 0:
return 1
else:
```

```
return n * factorial(n - 1)

def exp(x):
def exp_help(x, n):
z, l = rational(1, 1), rational(1, 1)
for i in range(1, n + 1):
z = z * x[n]
l = l + (z * rational(1, factorial(i)))
return l
return real(lambda n: exp_help(x, n))

class rational:
def __init__(self, p=0, q=1): self.p, self.q = p // euclid(p, q), q // euclid(p, q)
def __repr__(self): return "(" + str(self.p) + "," + str(self.q) + ")"
def __getitem__(self, key): return self

def __add__(self, other): return rational(self.p * other.q + other.p * self.q, self.q * other.q)
def __neg__(self): return rational(-self.p, self.q)
def __sub__(self, other): return (self + (-other))
def __mul__(self, other): return rational(self.p * other.p, self.q * other.q)
def __truediv__(self,other): return rational(self.p * other.q, self.q * other.p )

def __abs__(self): return rational(abs(self.p), abs(self.q))

def __lt__(self, other): return self.p * other.q < other.p * self.q
def __le__(self, other): return self.p * other.q <= other.p * self.q
def __eq__(self, other): return self.p * other.q == other.p * self.q
def __gt__(self, other): return self.p * other.q > other.p * self.q
def __ge__(self, other): return self.p * other.q >= other.p * self.q
def __ne__(self, other): return self.p * other.q != other.p * self.q

class real:
def __init__(self, rat=rational(0, 1)):
if type(rat) is rational:
self.appr = lambda n: rat
elif type(rat) is int:
self.appr = lambda n: rational(rat, 1)
else:
self.appr = rat

def __str__(self): return '{0:g}'.format(self[10].p / self[10].q)
def __repr__(self): return '{0:g}'.format(self[10].p / self[10].q)

def __getitem__(self, key): return self.appr(key)

def __add__(self, other): return real(lambda n: self[n+1] + other[n+1])
def __neg__(self): return real(lambda n: -self[n])
```

```
def __sub__(self, other): return self + (-other)
def __mul__(self, other):
print(self[0])
print(other[0])
l = abs(self[0]) + abs(other[0])
k = ((l.p // l.q) + 2).bit_length()
return real(lambda n: self[n + k] * other[n + k])


def __abs__(self): return real(lambda n: abs(self[n]))

class infreal:
def __init__(self, digits, k=0):
self.k = k
self.digits = digits

def __repr__(self):
if self.k == 0:
return "^" + ", ".join([str(self.digits(i)) for i in range(self.k,self.k+7)]) + ", ..."
else:
return ", ".join([str(self.digits(i)) for i in range(self.k)]) + ", ^" + ", ".join([str(self.digits(i)) for i in range(self.k,self.k+7)]) + ", ..."

def __getitem__(self, key): return self.digits(key)

def __add__(self, other):
k = max(self.k, other.k)
return infreal(lambda n: self.digits(n - (k-self.k)) + other.digits(n - (k-other.k)), k)
def __neg__(self): return infreal(lambda n: -self[n], self.k)
def __sub__(self, other): return (self + (-other))
def __mul__(self, other):
k = self.k + other.k
def digits(n):
if n < 0:
return zero
else:
return reduce((lambda x,y:x+y), [self.digits(i) * other.digits(n - i) for i in range(n + 1)])
return infreal(digits, k)

def __pow__(self,p):
temp = One
for i in range(0, p):
temp = temp * self
return temp

e = exp(rational(1, 1))
zero = real(0)
```

```
Zero = infreal(lambda n: zero, 10)
one = real(1)
minusone = real(-1)
One = infreal(lambda n: one if n==10 else zero, 10)
Twohalf = infreal(lambda n: real(rational(5,2)) if n==0 else zero,0)
phi = real(rational(22,7))
Omega = infreal(lambda n: one if n == 0 else zero, 1)
Epsilon = infreal(lambda n: one if n == 1 else zero, 0)
Funny = infreal(lambda n: real(rational(n + 1, 1)) * e if n > -1 else zero, 0)
```

Listing A.1: The complete code of the computable $\mathbb{R}^{\mathbb{Z}_<}$

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Glossary

$\mathfrak{e}$-balls	Balls with radius r, where r is an infinitesimal. 66
$\mathfrak{e}$-distinguishable	Two points are $\mathfrak{e}$-distinguishable when there is an $\mathfrak{e}$-open set containing precisely one of the two points. 68
$\mathfrak{e}$-open	A set is $\mathfrak{e}$-open if and only if $\forall x \in O$ $\exists \mathfrak{e} \in \overset{m}{\Delta}{}^{\downarrow}$ s.t. $B_x(\mathfrak{e}) \subseteq O$. 67
ψ-balls	Balls that are created when $\mathtt{d}_\psi$ is used as the metric. 66
k, n-continuous	A new notion of continuity in $\mathbb{R}^{\mathbb{Z}_<}$, which is more fine-grained than the classical notion. 102
$\mathtt{St}$-balls	Balls with r radius, where $r \in \mathbb{R}$. 66
$\mathtt{St}$-distinguishable	Two points are St-distinguishable when there is an St-open set containing precisely one of the two points. 67
$\mathtt{St}$-open	A set is St-open if and only if $\forall x \in O$ $\exists n \in \mathbb{N}$ s.t. $B_x\left(\frac{1}{n}\right) \subseteq O$. 67
ball	A topological neighbourhood of a point in a certain set, which is $\mathbb{R}^{\mathbb{Z}_<}$ in this work. 65

chunk and permeate	A paraconsistent inference strategy which was introduced by Brown and Priest in 2004. The explanation about this strategy can be seen in Section 3.2. 41
Dedekind cut	One way of constructing the set of real numbers, as described in Definition 2.8. 13
grossone	A new theory of infinities which was introduced by Sergeyev in 2008. He denotes the "first" infinity with ①. 57
hyper Cauchy sequence	A sequence which satisfies the hyper Cauchy criterion. 114
hyperreals	An extension of the set $\mathbb{R}$ by adding infinitesimals and infinities. . . . 17
incoherence	A system is incoherence if and only if it can prove anything, desired or not. 34
inconsistency	A system contains an inconsistency if and only if it has a contradiction in it. 34
infinitesimal fractal	A geometric object that exhibits some form of self-similarity in an infinitesimal scale. 95
infinitesimally topologically continuous	A new notion of topological continuity denoted by $\mathfrak{c}$-continuous. 108

metric	A metric on a set A is a function which denotes how "far" two elements in A are. This function has to satisfy the corresponding four conditions. The metric in $\mathbb{R}^{\mathbb{Z}_<}$ is denoted by $\mathrm{d}(\mathbf{x}, \mathbf{y})$. 64
microstable	A new property of a function defined in $\mathbb{R}^{\mathbb{Z}_<}$. 72
non-standard part	This term is used to denote the infinity or infinitesimal part of a number, either in $\mathbb{R}^{\mathbb{Z}}$ or $\mathbb{R}^{\mathbb{Z}_<}$. 46
paraconsistent logic	A logic which permit inference from inconsistent information without giving a global absurdity. 28
pseudo-metric	A kind of metric which satisfies only three out of the four conditions satisfied by metric. The pseudo-metric in $\mathbb{R}^{\mathbb{Z}_<}$ is denoted by $\mathrm{d}_\psi(\mathbf{x}, \mathbf{y})$. 64
rat-balls	Balls with radius $\frac{1}{n}$, where $n \in \mathbb{N}$. 66
standard part	This term is used to denote the "real" part of a number, either in $\mathbb{R}^{\mathbb{Z}}$ or $\mathbb{R}^{\mathbb{Z}_<}$. 46
standard topologically continuous	A new notion of topological continuity denoted by St-continuous. 108
topological metric space	A topological space equipped with its notion of metric. 68
topological space	A pair of set together with its topology. 68

topology	A collection of subsets from a set which satisfies the four conditions described in Definition **??**. 68
transfer principle	The one important principle which connect $\mathbb{R}$ and $^*\mathbb{R}$. This principle can be seen in Definition 2.35. 22